INDIVIDUAL TEST

S.-T YAU COLLEGE MATH CONTESTS 2012

Algebra and Number Theory

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Prove that the polynomial $x^6 + 30x^5 - 15x^3 + 6x - 120$ cannot be written as a product of two polynomials of rational coefficients and positive degrees.

2. Let $\mathbb{F}_p$ be the field of p-elements and $GL_n(\mathbb{F}_p)$ the group of invertible n by n matrices.

(1) Compute the order of $GL_n(\mathbb{F}_p)$.
(2) Find a Sylow p-subgroup of $GL_n(\mathbb{F}_p)$.
(3) Compute the number of Sylow p-subgroups.

3. Let V be a finite dimensional vector space over complex field $\mathbb{C}$ with a nondegenerate symmetric bilinear form $(,)$. Let

$$O(V) = \{g \in GL(V) | (gu, gv) = (u, v),\ u, v \in V\}$$

be the orthogonal group. Prove that fixed point subspace $(V \otimes_{\mathbb{C}} V)^{O(n)}$ is 1-dimensional.

4. Let $\mathfrak{D}$ be the ring consisting of all linear differential operators of finite order on $\mathbb{R}$ with polynomial coefficients, of the form

$$D = \sum_{i=0}^{n} a_i(x) \frac{d^i}{dx^i}$$

for some natural number $n \in \mathbb{N}$ and polynomials $a_0(x), \cdots, a_n(x) \in \mathbb{R}[x]$. This ring R operates naturally on $M := \mathbb{R}[x]$, making M a left $\mathfrak{D}$-module.

(1) (to warm up) Suppose that $b(x) \in \mathbb{R}[x]$ is a non-zero polynomial in M, and let $c(x)$ be any element in M. Show that there is an element $D \in \mathfrak{D}$ such that $D(b(x)) = c(x)$.
(2) Suppose that m is a positive integer, $b_1(x), \cdots, b_m(x)$ are m polynomials in M linearly independent over $\mathbb{R}$ and $c_1(x), \cdots, c_m(x)$ are m polynomials in M. Prove that there exists an element $D \in \mathfrak{D}$ such that $D(b_i(x)) = c_i(x)$ for $i = 1, \cdots, m$.

5. Let Λ be a lattice of $\mathbb{C}$, i.e., a subgroup generated by two $\mathbb{R}$-linear independent elements. Let R be the subring of $\mathbb{C}$ consists of elements α such that $\alpha\Lambda \subset \Lambda$. Let $R^\times$ denote the group of invertible elements in R.

(1) Show that either $R = \mathbb{Z}$ or have rank 2 over $\mathbb{Z}$.

(2) Let $n \geq 3$ be a positive integer and $(R/nR)^\times$ the group of invertible elements in the quotient R/nR. Show that the canonical group homomorphism
$$R^\times \to (R/nR)^\times$$
is injective.
(3) Find maximal size of $R^\times$.

6. Let V be a (possible) infinite dimensional vector space over $\mathbb{R}$ with a positive definite quadratic norm $\|\cdot\|$. Let A be an additive subgroup of V with following properties:
(1) $A/2A$ is finite;
(2) for any real number c the set
$$\{a \in A : \qquad \|a\| < c\}$$
is finite.

Prove that A is of finite rank over $\mathbb{Z}$.