INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Analysis and Differential Equations

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Compute the integral

$$\int_0^\infty \frac{x^p}{1+x^2}dx, -1 < p < 1.$$

2. Construct a one to one conformal mapping from the region

$$U = \{z \in \mathbb{C} | |z - \frac{i}{2}| < \frac{1}{2}\} / \{z | |z - \frac{i}{4}| < \frac{1}{4}\}$$

onto the unit disk.

3. Let $f(x)$ be a C^2 function on $\mathbb{R}$. Show that

$$\sup |f'(x)|^2 \leq 4 \sup |f(x)| \sup |f''(x)|.$$

4. Let $f(x)$ be a real measurable function defined on $[a, b]$. Let $n(y)$ be the number of solutions of the equation $f(x) = y$. Prove that $n(y)$ is a measurable function on $\mathbb{R}$.

5. For $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, let g in L^q. Consider the linear functional F on L^p given by: $F(f)$ is equal to the integral of fg. Show that $||F|| = ||g||_q$.

6. Let $\mathbb{R}^n_+ = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_n > 0\}$. Show that the formula

$$u(x) = \frac{2x_n}{n\alpha_n} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x - y|^n} dy, x \in \mathbb{R}^n_+$$

is a solution of the problem

$$\Delta u = 0, \text{ in } \mathbb{R}^n_+, u = g \text{ on } \partial \mathbb{R}^n_+,$$

where α_n is the volume of the unit n dimensional sphere.