INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied Math. and Computational Math.

Please solve 4 out of the following 5 problems.

1. In the numerical integration formula

$$\int_{-1}^{1} f(x)dx \approx af(-1) + bf(c), \tag{1}$$

if the constants a, b, c can be chosen arbitrarily, what is the highest degree k such that the formula is exact for all polynomials of degree up to k? Find the constants a, b, c for which the formula is exact for all polynomials of degree up to this k.

2. Here is the definition of a moving least square approximation of a function $f(x)$ near a point $\overline{x}$ given K points x_k around $\overline{x}$ in $\mathbb{R}$, $k \in [1, \cdots, K]$.

$$\min_{P_{\overline{x}} \in \Pi_m} \sum_{k=1}^{K} |P_{\overline{x}}(x_k) - f_k|^2 \tag{2}$$

where $f_k = f(x_k)$, Π_m is the space of polynomials of degree less or equal to m, i.e.

$$P_{\overline{x}}(x) = \mathbf{b}_{\overline{x}}(x)^T \mathbf{c}(\overline{x}),$$

$\mathbf{c}(\overline{x}) = [c_0, c_1, \cdots, c_m]^T$ is the coefficient vector to be determined by (2), $\mathbf{b}_{\overline{x}}(x)$ is the polynomial basis vector, $\mathbf{b}_{\overline{x}}(x) = [1, x - \overline{x}, (x - \overline{x})^2, \ldots, (x - \overline{x})^m]^T$.
Assume that there are $K > m$ different points x_k and $f(x)$ is smooth,
(a) prove that there is a unique solution $\overline{P}_{\overline{x}}(x)$ to (2)
(b) denote $h = \max_k \ |x_k - \overline{x}|$, prove

$$|c_i - \frac{1}{i!} f^{(i)}(\overline{x})| = C(f, i) h^{m+1-i}, \ i = 0, 1, \ldots, m,$$

where $f^{(i)}(\cdot)$ is the i-th derivative of f and $C(f, i)$ denote some constant depending on f, i.
(c) if $S = \{x_k | k = 1, 2, \ldots, K\}$ are symmetrically distributed around $\overline{x}$, that is, if $x_k \in S$ then $2\overline{x} - x_k \in S$, prove that

$$|c_i - \frac{1}{i!} f^{(i)}(\overline{x})| = C(f, i) h^{m+2-i}, \ i = 0, 1, \ldots, m,$$

for i $(\in \{0, 1, \cdots, m\})$ with the same parity of m.

3. Describe the forward-in-time and center-in-space finite difference scheme for the one-wave wave equation:

$$u_t + u_x = 0.$$

(i). Conduct the von Neumann stability analysis and comment on their stability property.
(ii). Under what condition on Δt and Δx would this scheme be stable and convergent?
(iii). How many ways you can modify this scheme to make it stable when the CFL condition is satisfied.

4. Let C and D in $\mathbb{C}^{n\times n}$ be Hermitian matrices. Denote their eigenvalues by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n,$$

respectively. Then it is known that

$$\sum_{i=1}^{n} (\lambda_i - \mu_i)^2 \leq \|C - D\|_F^2.$$

1) Let A and B be in $\mathbb{C}^{n\times n}$. Denote their singular values by

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \quad \text{and} \quad \tau_1 \geq \tau_2 \geq \cdots \geq \tau_n,$$

respectively. Prove that the following inequality holds:

$$\sum_{i=1}^{n} (\sigma_i - \tau_i)^2 \leq \|A - B\|_F^2.$$

2) Given $A \in \mathbb{R}^{n\times n}$ and its SVD is $A = U\Sigma V^T$, where $U = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$, $V = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ are orthogonal matrices, and

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

Suppose $\text{rank}(A) > k$ and denote by

$$U_k = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k), \quad V_k = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \quad \Sigma_k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k),$$

and

$$A_k = U_k \Sigma_k V_k^T = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T.$$

Prove that

$$\min_{\text{rank}(B)=k} \|A - B\|_F^2 = \|A - A_k\|_F^2 = \sum_{i=k+1}^{n} \sigma_i^2.$$

3) Let the vectors $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$, be in the space $\mathcal{W}$ with dimension d, where $d \ll n$. Let the orthonormal basis of $\mathcal{W}$ be $W \in \mathbb{R}^{n \times d}$. Then we can represent $\mathbf{x}_i$ by

$$\mathbf{x}_i = \mathbf{c} + W\mathbf{r}_i + \mathbf{e}_i, \ i = 1, 2, \ldots, n,$$

where $\mathbf{c} \in \mathbb{R}^n$ is a constant vector, $\mathbf{r}_i \in \mathbb{R}^d$ is the coordinate of the point $\mathbf{x}_i$ in the space $\mathcal{W}$, and $\mathbf{e}_i$ is the error. Denote $R = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$ and $E = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$. Find W, R and $\mathbf{c}$ such that the error $\|E\|_F$ is minimized.
(*Hint*: write $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = \mathbf{c}(1, 1, \ldots, 1) + WR + E$.)

5. Two primes p and q are called *twin primes* if $q = p + 2$. For example, 5 and 7, 11 and 13, 29 and 31 are twin primes. There is a still unproven (but extensively numerically verified) conjecture that there are infinitely many twin primes and that they are rather common. Show how to factor an integer N which is a product of two twin primes.