INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Geometry and Topology

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Show that $\pi_3(S^2) \neq 0$.

2. Let M be a smooth manifold of dimension n, and $X_1, \cdots, X_k$ be k everywhere linearly independent smooth vector fields on an open set $U \subset M$ satisfying that $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$. Prove that for any point $p \in U$ there is a coordinate chart (V, y^i) with $p \in V \subseteq U$ and coordinates $\{y^1, \cdots, y^n\}$ such that $X_i = \frac{\partial}{\partial y^i}$ on V for each $1 \leq i \leq k$.

3. Show that any self homeomorphism of $\mathbb{CP}^2$ is orientation preserving.

4. Prove the following version of the isoperimetric inequality: Suppose C is a simple (that is, without self-intersection), smooth, closed curve in the Euclidean plane, with length L. Show that the area enclosed by C is less than or equal to $\frac{L^2}{4\pi}$, and the equality occurs when and only when C is a round circle.

5. Let $x : M \to \mathbb{R}^3$ be a closed surface in 3-dimensional Euclidean space. Its Gaussian curvature and mean curvature are denoted by K and H respectively. Prove that:

$$\iint_M H dA + \iint_M pK dA = 0, \quad \iint_M pH dA + \iint_M dA = 0,$$

where $p = \vec{x} \cdot \vec{n}$ is the support function of M, $\vec{x}$ denotes the position vector of M, $\vec{n}$ denotes the unit normal to M, and dA is the area element of M.

6. Write the structure equation of an orthonormal frame on a Riemannian manifold. Prove the following Riemannian metric g has constant sectional curvature c using the structure equation:

$$g = \frac{\sum_{i=1}^n (dx^i)^2}{[1 + \frac{c}{4} \sum_{i=1}^n (x^i)^2]^2}$$

where $(x^1, \ldots, x^n)$ is a local coordinate system.