INDIVIDUAL TEST  
S.-T YAU COLLEGE MATH CONTESTS 2012  

Geometry and Topology  

Please solve 5 out of the following 6 problems, or highest scores of 5 problems will be counted.

1. Show that $\pi_3(S^2) \neq 0$.

2. Let $M$ be a smooth manifold of dimension $n$, and $X_1, \cdots, X_k$ be $k$ everywhere linearly independent smooth vector fields on an open set $U \subset M$ satisfying that $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$. Prove that for any point $p \in U$ there is a coordinate chart $(V, y^i)$ with $p \in V \subset U$ and coordinates $\{y^1, \cdots, y^n\}$ such that $X_i = \frac{\partial}{\partial y^i}$ on $V$ for each $1 \leq i \leq k$.

3. Show that any self homeomorphism of $\mathbb{CP}^2$ is orientation preserving.

4. Prove the following version of the isoperimetric inequality: Suppose $C$ is a simple (that is, without self-intersection), smooth, closed curve in the Euclidean plane, with length $L$. Show that the area enclosed by $C$ is less than or equal to $\frac{L^2}{4\pi}$, and the equality occurs when and only when $C$ is a round circle.

5. Let $x : M \to \mathbb{R}^3$ be a closed surface in 3-dimensional Euclidean space. Its Gaussian curvature and mean curvature are denoted by $K$ and $H$ respectively. Prove that:

\[
\int_M H dA + \int_M pK dA = 0, \quad \int_M pH dA + \int_M dA = 0,
\]

where $p = \vec{x} \cdot \vec{n}$ is the support function of $M$, $\vec{x}$ denotes the position vector of $M$, $\vec{n}$ denotes the unit normal to $M$, and $dA$ is the area element of $M$.

6. Write the structure equation of an orthonormal frame on a Riemannian manifold. Prove the following Riemannian metric $g$ has constant sectional curvature $c$ using the structure equation:

\[
g = \frac{\sum_{i=1}^n (dx^i)^2}{\left[1 + \frac{c}{4} \sum_{i=1}^n (x^i)^2\right]^2}
\]

where $(x^1, \ldots, x^n)$ is a local coordinate system.