On a counterexample to a problem in 2012 S.T. Yau College Math Contests

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The following problem is the 4th problem of the individual test on analysis and differential equations in the 2012 S.T. Yau College Math Contests:

Problem. *Let $f(x)$ be a real measurable function defined on $[a, b]$. Let $n(y)$ be the number of solutions of the equation $f(x) = y$. Prove that $n(y)$is a measureable function on $\mathbb{R}$.*

However, it seems that there is a counterexample to this problem.

Let $a = 0, b = 1$. Consider the Cantor set $C \subseteq [0, 1]$. Then C is a set of measure 0 and there is a one to one map from C to $\mathbb{R}$.

Let $A \subseteq \mathbb{R}$ be a unmeasureable set such that $0 \notin A$. Then there is a set $A' \subseteq C$ which is in one to one correspondence with A. Let

$$\phi : A' \to A$$

be a one to one map from A' to A.

Then we can construct a function f as follows:

$$\begin{aligned} f: \quad [0,1] \quad &\to \quad \mathbb{R} \\ x \quad &\mapsto \begin{cases} 0 & \text{if } x \notin A' \\ \phi(x) & \text{if } x \in A' \end{cases} \end{aligned}$$

Since $A' \subseteq C$, A' is a set of measure zero, hence $f = 0$ on a.e. $[0, 1]$. So f is a measureable function. However, the function $n(y)$ associated to f is:

$$n(y) = \begin{cases} 0 & \text{if } y \notin A \cup \{0\} \\ \infty & \text{if } y = 0 \\ 1 & \text{if } y \in A \end{cases}$$

Since A is unmeasureable, it is clear that $n(y)$ is an unmeasureable function.

Even if we require that $n(y)$ is everywhere finite on $\mathbb{R}$, we can still construct similar counterexamples. For instance, we can find a unmeasurable set $A \subseteq [0,1]$, and choose a subset A' in the Cantor set C such that there is a one to one correspondence ϕ from A' to A. Then define f as follows:

$$\begin{aligned} f: \quad [0,1] &\rightarrow \qquad \mathbb{R} \\ x &\mapsto \begin{cases} x+2 & \text{if } x \notin A' \\ \phi(x) & \text{if } x \in A' \end{cases} \end{aligned}$$

Since $f = x + 2$ on a.e. $[0,1]$, f is measureable.

The associated $n(y)$ is:

$$n(y) = \begin{cases} 1 & \text{if } y \in A \text{ or } y \in [2,3] \backslash (A' + \{2\}) \\ 0 & \text{otherwise} \end{cases}$$

Since $A \subseteq [0,1]$ is an unmeasureable set, A' is a set of zero measure, $n(y)$ is everywhere finite and unmeasureable.