The 6$^{\text{th}}$ Romanian Master of Mathematics Competition

Solutions for the Day 1

Problem 1. For a positive integer a, define a sequence of integers $x_1, x_2, \ldots$ by letting $x_1 = a$ and $x_{n+1} = 2x_n + 1$ for $n \geq 1$. Let $y_n = 2^{x_n} - 1$. Determine the largest possible k such that, for some positive integer a, the numbers $y_1, \ldots, y_k$ are all prime.

(Russia) Valery Senderov

Solution. The largest such is $k = 2$. Notice first that if y_i is prime, then x_i is prime as well. Actually, if $x_i = 1$ then $y_i = 1$ which is not prime, and if $x_i = mn$ for integer $m, n > 1$ then $2^m - 1 \mid 2^{x_i} - 1 = y_i$, so y_i is composite. In particular, if $y_1, y_2, \ldots, y_k$ are primes for some $k \geq 1$ then $a = x_1$ is also prime.

Now we claim that for every odd prime a at least one of the numbers y_1, y_2, y_3 is composite (and thus $k < 3$). Assume, to the contrary, that y_1, y_2, and y_3 are primes; then x_1, x_2, x_3 are primes as well. Since $x_1 \geq 3$ is odd, we have $x_2 > 3$ and $x_2 \equiv 3 \pmod 4$; consequently, $x_3 \equiv 7 \pmod 8$. This implies that 2 is a quadratic residue modulo $p = x_3$, so $2 \equiv s^2 \pmod p$ for some integer s, and hence $2^{x_2} = 2^{(p-1)/2} \equiv s^{p-1} \equiv 1 \pmod p$. This means that $p \mid y_2$, thus $2^{x_2} - 1 = x_3 = 2x_2 + 1$. But it is easy to show that $2^t - 1 > 2t + 1$ for all integer $t > 3$. A contradiction.

Finally, if $a = 2$, then the numbers $y_1 = 3$ and $y_2 = 31$ are primes, while $y_3 = 2^{11} - 1$ is divisible by 23; in this case we may choose $k = 2$ but not $k = 3$.

Remark. The fact that $23 \mid 2^{11} - 1$ can be shown along the lines in the solution, since 2 is a quadratic residue modulo $x_4 = 23$.

Problem 2. Does there exist a pair (g, h) of functions $g, h\colon \mathbb{R} \to \mathbb{R}$ such that the only function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbb{R}$ is the identity function $f(x) \equiv x$?

(United Kingdom) Alexander Betts

Solution 1. Such a tester pair exists. We may biject $\mathbb{R}$ with the closed unit interval, so it suffices to find a tester pair for that instead. We give an explicit example: take some positive real numbers α, β (which we will specify further later). Take

$$g(x) = \max(x - \alpha, 0) \qquad \text{and} \qquad h(x) = \min(x + \beta, 1).$$

Say a set $S \subseteq [0, 1]$ is *invariant* if $f(S) \subseteq S$ for all functions f commuting with both g and h. Note that intersections and unions of invariant sets are invariant. Preimages of invariant sets under g and h are also invariant; indeed, if S is invariant and, say, $T = g^{-1}(S)$, then $g(f(T)) = f(g(T)) \subseteq f(S) \subseteq S$, thus $f(T) \subseteq T$.

We claim that (if we choose $\alpha + \beta < 1$) the intervals $[0, n\alpha - m\beta]$ are invariant where n and m are nonnegative integers with $0 \le n\alpha - m\beta \le 1$. We prove this by induction on $m + n$.

The set $\{0\}$ is invariant, as for any f commuting with g we have $g(f(0)) = f(g(0)) = f(0)$, so $f(0)$ is a fixed point of g. This gives that $f(0) = 0$, thus the induction base is established.

Suppose now we have some m, n such that $[0, n'\alpha - m'\beta]$ is invariant whenever $m' + n' < m + n$. At least one of the numbers $(n-1)\alpha - m\beta$ and $n\alpha - (m-1)\beta$ lies in $(0, 1)$. Note however that in the first case $[0, n\alpha - m\beta] = g^{-1}\left([0, (n-1)\alpha - m\beta]\right)$, so $[0, n\alpha - m\beta]$ is invariant. In the second case $[0, n\alpha - m\beta] = h^{-1}\left([0, n\alpha - (m-1)\beta]\right)$, so again $[0, n\alpha - m\beta]$ is invariant. This completes the induction.

We claim that if we choose $\alpha + \beta < 1$, where $0 < \alpha \notin \mathbb{Q}$ and $\beta = 1/k$ for some integer $k > 1$, then all intervals $[0, \delta]$ are invariant for $0 \le \delta < 1$. This occurs, as by the previous claim, for all nonnegative integers n we have $[0, (n\alpha \bmod 1)]$ is invariant. The set of $n\alpha \bmod 1$ is dense in $[0, 1]$, so in particular

$$[0, \delta] = \bigcap_{(n\alpha \bmod 1) > \delta} [0, (n\alpha \bmod 1)]$$

is invariant.

A similar argument establishes that $[\delta, 1]$ is invariant, so by intersecting these $\{\delta\}$ is invariant for $0 < \delta < 1$. Yet we also have $\{0\}, \{1\}$ both invariant, which proves f to be the identity.

Solution 2. Let us agree that a sequence $\mathbf{x} = (x_n)_{n=1,2,\ldots}$ is *cofinally non-constant* if for every index m there exists an index $n > m$ such that $x_m \neq x_n$.

Biject $\mathbb{R}$ with the set of cofinally non-constant sequences of 0's and 1's, and define g and h by

$$g(\epsilon, \mathbf{x}) = \begin{cases} \epsilon, \mathbf{x} & \text{if } \epsilon = 0 \\ \mathbf{x} & \text{else} \end{cases} \qquad \text{and} \qquad h(\epsilon, \mathbf{x}) = \begin{cases} \epsilon, \mathbf{x} & \text{if } \epsilon = 1 \\ \mathbf{x} & \text{else} \end{cases}$$

where $\epsilon, \mathbf{x}$ denotes the sequence formed by appending $\mathbf{x}$ to the single-element sequence ϵ. Note that g fixes precisely those sequences beginning with 0, and h fixes precisely those beginning with 1.

Now assume that f commutes with both f and g. To prove that $f(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ we show that $\mathbf{x}$ and $f(\mathbf{x})$ share the same first n terms, by induction on n.

The base case $n = 1$ is simple, as we have noticed above that the set of sequences beginning with a 0 is precisely the set of g-fixed points, so is preserved by f, and similarly for the set of sequences starting with 1.

Suppose that $f(\mathbf{x})$ and $\mathbf{x}$ agree for the first n terms, whatever $\mathbf{x}$. Consider any sequence, and write it as $\mathbf{x} = \epsilon, \mathbf{y}$. Without loss of generality, we may (and will) assume that $\epsilon = 0$, so $f(\mathbf{x}) = 0, \mathbf{y}'$ by the base case. Yet then $f(\mathbf{y}) = f(h(\mathbf{x})) = h(f(\mathbf{x})) = h(0, \mathbf{y}') = \mathbf{y}'$. Consequently, $f(\mathbf{x}) = 0, f(\mathbf{y})$, so $f(\mathbf{x})$ and $\mathbf{x}$ agree for the first $n+1$ terms by the inductive hypothesis.

Thus f fixes all of cofinally non-constant sequences, and the conclusion follows.

Solution 3. (*Ilya Bogdanov*) We will show that there exists a tester pair of *bijective* functions g and h.

First of all, let us find out when a pair of functions is a tester pair. Let $g, h \colon \mathbb{R} \to \mathbb{R}$ be arbitrary functions. We construct a directed graph $G_{g,h}$ with $\mathbb{R}$ as the set of vertices, its edges being painted with two colors: for every vertex $x \in \mathbb{R}$, we introduce a red edge $x \to g(x)$ and a blue edge $x \to h(x)$.

Now, assume that the function $f \colon \mathbb{R} \to \mathbb{R}$ satisfies $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbb{R}$. This means exactly that if there exists an edge $x \to y$, then there also exists an edge $f(x) \to f(y)$ of the same color; that is — f is an *endomorphism* of $G_{g,h}$.

Thus, a pair (g, h) is a tester pair if and only if the graph $G_{g,h}$ admits no nontrivial endomorphisms. Notice that each endomorphism maps a component into a component. Thus, to construct a tester pair, it suffices to construct a continuum of components with no nontrivial endomorphisms and no homomorphisms from one to another. It can be done in many ways; below we present one of them.

Let $g(x) = x + 1$; the construction of h is more involved. For every $x \in [0, 1)$ we define the set $S_x = x + \mathbb{Z}$; the sets S_x will be exactly the components of $G_{g,h}$. Now we will construct these components.

Let us fix any $x \in [0, 1)$; let $x = 0.x_1x_2\ldots$ be the binary representation of x. Define $h(x - n) = x - n + 1$ for every $n > 3$. Next, let $h(x - 3) = x$, $h(x) = x - 2$, $h(x - 2) = x - 1$, and $h(x - 1) = x + 1$ (that would be a "marker" which fixes a point in our component).

Next, for every $i = 1, 2, \ldots$, we define

(1) $h(x + 3i - 2) = x + 3i - 1$, $h(x + 3i - 1) = x + 3i$, and $h(x + 3i) = x + 3i + 1$, if $x_i = 0$;

(2) $h(x + 3i - 2) = x + 3i$, $h(x + 3i) = 3i - 1$, and $h(x + 3i - 1) = x + 3i + 1$, if $x_i = 1$.

Clearly, h is a bijection mapping each S_x to itself. Now we claim that the graph $G_{g,h}$ satisfies the desired conditions.

Consider any homomorphism $f_x \colon S_x \to S_y$ (x and y may coincide). Since g is a bijection, consideration of the red edges shows that $f_x(x + n) = x + n + k$ for a fixed real k. Next, there exists a blue edge $(x - 3) \to x$, and the only blue edge of the form $(y + m - 3) \to (y + m)$ is $(y - 3) \to y$; thus $f_x(x) = y$, and $k = 0$.

Next, if $x_i = 0$ then there exists a blue edge $(x + 3i - 2) \to (x + 3i - 1)$; then the edge $(y + 3i - 2) \to (y + 3i - 1)$ also should exist, so $y_i = 0$. Analogously, if $x_i = 1$ then there exists a blue edge $(x + 3i - 2) \to (x + 3i)$; then the edge $(y + 3i - 2) \to (y + 3i)$ also should exist, so $y_i = 1$. We conclude that $x = y$, and f_x is the identity mapping, as required.

Remark. If g and h are injections, then the components of $G_{g,h}$ are at most countable. So the set of possible pairwise non-isomorphic such components is continual; hence there is no bijective tester pair for a hyper-continual set instead of $\mathbb{R}$.

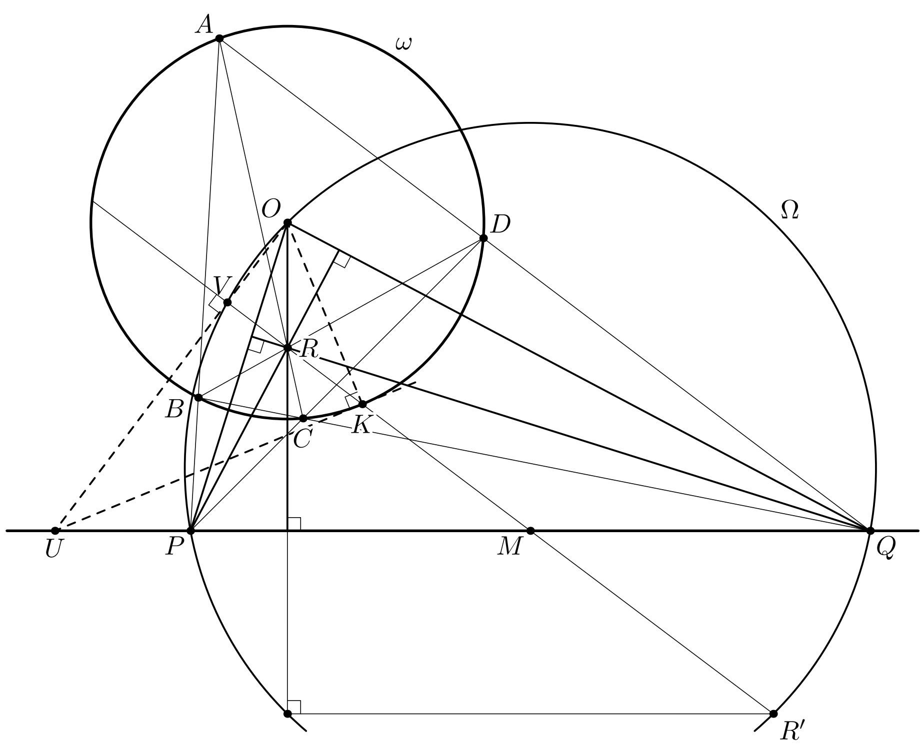
Problem 3. Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ, and let K be the common point of the segment MR and the circle ω. Prove that the circumcircle of the triangle KPQ and ω are tangent to one another.

(Russia) Medeubek Kungozhin

Solution. Let O be the centre of ω. Notice that the points P, Q, and R are the poles (with respect to ω) of the lines QR, RP, and PQ, respectively. Hence we have $OP \perp QR$, $OQ \perp RP$, and $OR \perp PQ$, thus R is the orthocentre of the triangle OPQ. Now, if $MR \perp PQ$, then the points P and Q are the reflections of one another in the line $MR = MO$, and the triangle PQK is symmetrical with respect to this line. In this case the statement of the problem is trivial.

Otherwise, let V be the foot of the perpendicular from O to MR, and let U be the common point of the lines OV and PQ. Since U lies on the polar line of R and $OU \perp MR$, we obtain that U is the pole of MR. Therefore, the line UK is tangent to ω. Hence it is enough to prove that $UK^2 = UP \cdot UQ$, since this relation implies that UK is also tangent to the circle KPQ.

From the rectangular triangle OKU, we get $UK^2 = UV \cdot UO$. Let Ω be the circumcircle of triangle OPQ, and let R' be the reflection of its orthocentre R in the midpoint M of the side PQ. It is well known that R' is the point of Ω opposite to O, hence OR' is the diameter of Ω. Finally, since $\angle OVR' = 90°$, the point V also lies on Ω, hence $UP \cdot UQ = UV \cdot UO = UK^2$, as required.



Remark. The statement of the problem is still true if K is the other common point of the line MR and ω.