S.-T. Yau College Student Mathematics Contests 2013

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Suppose that f is an integrable function on $\mathbf{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x||f(x)| > \alpha\}$. Prove that:

$$\int_{\mathbf{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$

2. Let $p(z)$ be a polynomial of degree $d \geq 2$, with distinct roots $a_1, a_2, \cdots, a_d$. Show that

$$\sum_{i=1}^{d} \frac{1}{p'(a_i)} = 0.$$

3. Let α be a number such that α/π is not a rational number. Show that:

1) $\lim_{N\to\infty} \Sigma_{n=1}^{N} e^{ik(x+n\alpha)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}dt.$

2) For every continuous periodic function $f : \mathbf{R} \to \mathbf{C}$ of period 2π, we have

$$\lim_{N\to\infty} \frac{1}{N}\Sigma_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt.$$

4. Let u be a positive harmonic function over the punctured complex plane $\mathbf{C}/\{0\}$. Show that u must be a constant function.

5. Suppose $H = L^2(B), B$ is the unit ball in $\mathbf{R}^d$. Let $K(x,y)$ be a measurable function on $B \times B$ that satisfies

$$|K(x,y)| \leq A|x-y|^{-d+\alpha}$$

for some $\alpha > 0$, whenever $x, y \in B$. Define

$$Tf(x) = \int_B K(x,y)f(y)dy$$

(a) Prove that T is a bounded operator on H.
(b) Prove that T is compact.

6. Let A be a $n\times n$ real non-degenerate symmetric matrix. For $\lambda \in \mathbf{R}^+$, we define: $\int_{\mathbf{R}} \exp(i\lambda x^2)dx = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \exp(i\lambda x^2 - \frac{1}{2}\epsilon x^2)dx$. Show that:

$$\int_{\mathbf{R}^n} \exp(i\frac{\lambda}{2} < Ax, x > -i < x, \xi >)dx$$

$$= (\frac{2\pi}{\lambda})^{n/2} |\det(A)|^{-1/2} \exp(-\frac{i}{2\lambda} < A^{-1}\xi, \xi >) \exp(\frac{i\pi}{4} sgnA).$$

Here $\lambda \in \mathbf{R}^+, \xi \in \mathbf{R}^n, sgn(A) = \nu_+(A) - \nu_-(A), \nu_+(A)(\nu_-(A))$ is the number of positive (negative) eigenvalues of A.