S.-T. Yau College Student Mathematics Contests 2013

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Find the homology and fundamental group of the space $X = S^1 \times S^1/\{p, q\}$ obtained from the torus by identifying two distinct points p, q to one point.

2. Suppose (X, d) is a compact metric space and $f : X \to X$ is a map so that $d(f(x), f(y)) = d(x, y)$ for all x, $y \in X$. Show that f is an onto map.

3. Let M^2 ba a complete regular surface and K be the Gaussian curvature. Suppose $\sigma : [0, \infty) \to M$ is a geodesic such that $K(\sigma(t)) \leq f(t)$, where f is a differentiable function on $[0, \infty)$. Prove that any solution $u(t)$ of the equation

$$u''(t) + f(t)u(t) = 0$$

has a zero on $[0, t_0]$, where $\sigma(t_0)$ is the first conjugate point to $\sigma(0)$ along σ.

4. Let g_1, g_2 be Riemannian metrics on a differentiable manifold M, and denote by R_1 and R_2 their respective Riemannian curvature tensor. Suppose that $R_1(X, Y, Y, X) = R_2(X, Y, Y, X)$ holds for any tangent vectors X, $Y \in T_pM$. Show that $R_1(X, Y, Z, W) = R_2(X, Y, Z, W)$ for any X, Y, Z, $W \in T_pM$.

5. Let M^n be an even dimensional, orientable Riemannian manifold with positive sectional curvature. Let $\sigma : [0, l] \to M$ be a closed geodesic, namely, σ is a geodesic with $\sigma(0) = \sigma(l)$ and $\sigma'(0) = \sigma'(l)$. Show that there exist an $\epsilon > 0$ and a smooth map $F : [0, l] \times (-\epsilon, \epsilon) \to M$, such that $F(t, 0) = \sigma(t)$, and for any fixed $s \neq 0$ in $(-\epsilon, \epsilon)$, $\sigma_s(t) = F(t, s)$ is a closed smooth curve with length less than that of σ.

6. Let (M^2, ds^2) be a minimal surface in $\mathbb{R}^3$, where ds^2 is the restriction of the Euclidean metric. Assume that the Gaussian curvature K of (M^2, ds^2) is negative. Denote by $\widetilde{K}$ the Gaussian curvature of the metric $\widetilde{ds^2} = -K ds^2$. Show that $\widetilde{K} = 1$.