S.-T. Yau College Student Mathematics Contests 2013

Probability and Statistics Problems

Individual

Please solve 5 out of the following 6 problems.

Problem 1. Let (X_n) be a sequence of random variables.

(1) Assume that $\sum_{n=0}^{\infty} P(|X_n| > n) < \infty$. Prove that $\limsup_{n\to\infty} \frac{|X_n|}{n} \leq 1$.

(2) Prove that (X_n) converges in probability to 0 if and only if for certain $r > 0$, $E\left[\frac{|X_n|^r}{1+|X_n|^r}\right] \to 0$.

Problem 2. Let X and Y be independent $N(0,1)$ random variables.

(1) Find $E[X+Y|X \geq 0, Y \geq 0]$;

(2) Find the distribution function of $X+Y$ given that $X \geq 0$ and $Y \geq 0$.

(Hint: For b) using the fact that $U = (X+Y)/\sqrt{2}$ and $V = (X-Y)/\sqrt{2}$ are independent and $N(0,1)$ distributed.)

Problem 3. Let $\{X_n\}$ be a sequence of independent and identically distributed continuous real valued random variables, and regard n as time. Let A_n be the following event:

$$A_n = \{X_n = \max\{X_1, X_2, \cdots, X_n\}\}.$$

We say that a maximum record occurs at n in such an event.

(1) Evaluate the probability $P(A_n)$.

(2) Denote by Y_n the number of maximum records occurred until time n, i.e.,

$$Y_n = \text{the number of } \ \{1 \leq k \leq n : X_k = \max\{X_1, X_2, \cdots, X_k\}\}.$$

Evaluate the expectation EY_n and the variance DY_n.

Problem 4. Let $X = (X_1, \cdots, X_n)$ be an iid sample from an exponential density with mean θ. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$. Let $P(X)$ = your p-value for an appropriate test.

(a) What is $E_{\theta_0}(P(X))$? Derive your answer explicitly.

(b) Derive $E_\theta(P(X))$ for $\theta \neq \theta_0$. Specifically, assuming only one sample, i.e. $n = 1$, calculate $E_\theta(P(X))$ as explicitly as possible for $\theta \neq \theta_0$.

(c) When there is only one sample, is $E_\theta(P(X))$ a decreasing function of θ? In general, can you prove your result for an arbitrary MLR family?

Problem 5. Let X_1, X_2 be iid uniform on $\theta - \frac{1}{2}$ to $\theta + \frac{1}{2}$.

(a) Show that for any given $0 < \alpha < 1$, you can find $c > 0$ such that

$$P_\theta\{\bar{X} - c < \theta < \bar{X} + c\} = 1 - \alpha,$$

where $\bar{X}$ is the sample mean.

(b) Show that for ϵ positive and sufficiently small

$$P_\theta\{\bar{X} - c < \theta < \bar{X} + c \mid |X_2 - X_1| \geq 1 - \epsilon\} = 1$$

(c) The statement in (a) is used to assert that $\bar{X} \pm c$ is a $100(1-\alpha)\%$ confidence interval for θ. Does the assertion in (b) contradict this? If your sample observations are $X_1 = 1, X_2 = 2$, would you use the confidence interval in (a)?

Problem 6. Suppose you want to estimate the total number of enemy tanks in a war on the basis of the captured tanks. Assume without loss of generality that the tank serial numbers are $1, 2, \cdots, N$, where N is the unknown total number of enemy tanks. Also assume the serial numbers of the n captured tanks are iid uniform on $1, 2, \cdots, N$. (This is a simplified assumption which provides a good approximation if $n << N$).

(a) Find the complete sufficient statistic.

(b) Suggest how you may find the minimum variance unbiased estimate of N.