S.-T. Yau College Student Mathematics Contests 2013

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Suppose $\Delta = \{z \in \mathbf{C} \mid |z| < 1\}$ is the open unit disk in the complex plane. Show that for any holomorphic function $f : \Delta \to \Delta$,

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \tag{1}$$

for all z in Δ. If equality holds in (1) for some $z_0 \in \Delta$, show that $f \in \mathrm{Aut}(\Delta)$, and that

$$\frac{|f'(z)|}{1-|f(z)|^2} = \frac{1}{1-|z|^2}$$

for all $z \in \Delta$.

2. Let f be a function of bounded variation on $[a,b]$, f_1 its generalized derivative as a measure, i.e. $f(x)-f(a) = \int_a^x f_1(y)dy$ for every $x \in [a,b]$ and $f_1(x)$ is an integrable function on $[a,b]$. Let f' be its weak derivative as a generalized function, i.e. $\int_a^b f(x)g'(x)dx = -\int_a^b f'(x)g(x)dx$, for any smooth function $g(x)$ on $[a,b], g(a) = g(b) = 0$. Show that:

a) If f is absolutely continuous, then $f' = f_1$.

b) If the weak derivative f' of f is an integrable function on $[a,b]$, then $f(x)$ is equal to an absolutely continuous function outside a set of measure zero.

3. Show that the convex hull of the roots of any polynomial contains all its critical points as well as all the zeros of higher derivatives of the polynomial. Here the convex hull of a given bounded set in the plane is the smallest convex set containing the given set in the plane.

4. Let $D \subset \mathbf{R}^3$ be an open domain. Show that every smooth vector field $\mathbf{F} = (P,Q,R)$ over D can be written as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ such that $rot(\mathbf{F}_1) = 0, div(\mathbf{F}_2) = 0$, where $rot(\mathbf{F}) = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}), div(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

5. Let $\mathbf{H}$ be a Hilbert space and $\mathbf{A}$ a compact self-adjoint linear operator over $\mathbf{H}$. Show that there exists an orthor-normal basis of $\mathbf{H}$ consisting of eigenvectors φ_n of $\mathbf{A}$ with non-zero eigenvalues λ_n such that every vector $\xi \in \mathbf{H}$ can be written as: $\xi = \Sigma_k c_k \varphi_k + \xi'$, where $\xi' \in Ker\mathbf{A}, i.e., \mathbf{A}\xi' = 0$. We also have $\mathbf{A}\xi = \Sigma_k \lambda_k c_k \varphi_k$.

If there are infinitely many eigenvectors then $\lim_{n\to\infty} \lambda_n = 0$.

6. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x')$$

for $0 \leq \lambda \leq 1$ and each $x, x' \in \mathbb{R}$, and is called strictly convex if

$$f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x')$$

for $0 < \lambda < 1$. We assume that $|f(x)| < \infty$ whenever $|x| < \infty$.

(a) Show that a convex function f is continuous and the function

$$g(y) = \max_{x \in \mathbf{R}} (xy - f(x))$$

is a well-defined convex function over $\mathbf{R}$.

(b) Show that a convex function f is differentiable except at most countably many points.

(c) f is differentiable everywhere if both f and g are strictly convex.

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Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let X be the space

$$\{(x, y, 0) \mid x^2 + y^2 = 1\} \cup \{(x, 0, z) \mid x^2 + z^2 = 1\}$$

Find the fundamental group $\pi_1(\mathbb{R}^3 \setminus X)$.

2. Let M be a smooth connected manifold and $f : M \to M$ be an injective smooth map such that $f \circ f = f$. Show that the image set $f(M)$ is a smooth submanifold in M.

3. Let $T^2 = \{(z, w) \in \mathbb{C}^2 \mid \ |z| = 1, |w| = 1\}$ be the torus. Define a map $f : T^2 \to T^2$ by $f(z, w) = (zw^3, w)$. Prove that f is a diffeomorphism.

4. Prove: Any 3-dimensional Einstein manifold has constant curvature.

5. State and prove the Myers theorem for complete Riemannian manifolds.

6. Let C be a regular closed curve in $\mathbb{R}^3$. Its torsion is τ. The integral $\frac{1}{2\pi}\int_C \tau ds$ is called the total torsion of C, where s is the arc length parameter. Prove: Given a smooth surface M in $\mathbb{R}^3$, if for any regular closed curve C on M, the total torsion of C is always an integer, then M is a part of a sphere or a plane.

S.-T. Yau College Student Mathematics Contests 2013

Algebra and Number Theory

Team

The exam contains 6 problems. Please choose 5 of them to work on.

1. (20pt) Let A be an $n \times n$ skew symmetric real matrix.

1.1 (10 pt) Prove that all eigenvalues of A are imaginary or zero and that e^A is orthogonal.

1.2 (10 pt) Find conditions on an orthogonal B such that $B = e^A$ is solvable for some skew symmetric and real matrix A.

2. (20pt) Let E/F be a field extension. Let A be an $m \times m$ matrix with entries in E such that $tr(A^n)$ belongs to F for every $n \geq 2$. Show that $tr(A)$ belongs to F by following steps.

2.1 (5pt) Show that there is a polynomial $P(x) = \sum_i a_i x^i \in \bar{E}[x]$ with $a_0 = 1$ such that

$$\sum_i a_i tr(A^{i+k}) = 0, \qquad \forall k \geq 1.$$

2.2 (5pt) Show that we have a polynomial $Q = \sum_i b_i x^i \in F[x]$ with $b_0 = 1$ such that

$$\sum_i b_i tr(A^{i+k}) = 0, \qquad \forall k \geq 2.$$

2.3 (5pt) Let $t \in \bar{E}$ be an eigenvalue of A with multiplicity m invertible in F. Show that $Q(t) = 0$.

2.4 (5pt) Show that $tr(A)$ belongs to F.

Hint: Let $t_i \in \bar{E}$ be all distinct non-zero eigen values of A with multiplicity m_i invertible in F. Then

$$tr(A^n) = \sum_i m_i t_i^n.$$

3. (20pt) Let p be a prime and $G = \mathrm{SL}_2(\mathbb{F}_p)$.

3.1 (10pt) Find the order of G.

3.2 (10pt) Show that the order of every element of G divides either $(p^2 - 1)$ or $2p$.

4. (20pt) Let S_4 be the symmetric group of 4 letters.

4.1 (10pt) Classify all complex irreducible representations of S_4;

4.2 (10pt) Find the character table of S_4.

5. (20pt) Let $\mathbb{F}_2$ be the finite field of two elements.

5.1 (10pt) Find all irreducible polynomials of degree 2 and 3 over $\mathbb{F}_2$;

5.2 (10pt) What is the number of irreducible polynomials of degree 6 over $\mathbb{F}_2$?

6. (20pt) Let F be the splitting field of $x^4 - 2$.

6.1 (10pt) Describe the field F and the Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$.

6.2 (10pt) Describe all subfields K of F and corresponding Galois subgroups $G_K = \mathrm{Gal}(F/K)$.

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Applied Math. and Computational Math.

Team

Please solve as many problems as you can!

1. Scaling behavior is one of the most important phenomena in scientific modeling and mathematical analysis. The following problem shows the universality and rigidity of scaling limits.

(a): (10 points) Suppose $U > 0$ is an increasing function on $[0, \infty)$ and there is a function $0 < \psi(x) < \infty$ for $x > 0$ such that

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \psi(x), \quad \text{for all } x > 0$$

Then $\psi(x) = x^\alpha$ for some $\alpha \geq 0$.

(b): (10 points) The above problem can be generalized as:

Suppose $U > 0$ is an increasing function on $[0, \infty)$ and there is an extended function $0 \leq \psi(x) \leq \infty$ and a set A dense in $[0, \infty)$ such that

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \psi(x), \quad \text{for all } x \in A$$

Then $\psi(x) = x^\alpha$ for some $\alpha \in [0, \infty]$.

(c): (15 points) (Warming: this part is hard).

A function $L : (0, \infty) \to (0, \infty)$ is called slowly varying at ∞ if

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1, \qquad \text{for all } x \in A \text{ dense in } (0, \infty)$$

The function U in **(a)** and **(b)** can be recast as $U(x) = c\,x^\alpha L(x)$ for some $c \geq 0$. Now we can extend **(b)** to an even more general setting:

Suppose $U > 0$ is increasing on $(0, \infty)$, set A dense in $[0, \infty)$ and

$$\lim_{n\to\infty} a_n U(b_n x) = \psi(x) \leq \infty, \text{ for all } x \in A.$$

where $b_n \to \infty$ and $\frac{a_{n+1}}{a_n} \to 1$ for some interval. Then there is a real number $\alpha \in [0, \infty]$, constant $c \geq 0$, and a function L slowly varying at ∞ such that $\psi(x) = x^\alpha$ and $U(x) = c\,x^\alpha L(x)$.

2. The following three operators are important for many mathematics and physics problems. Let $\phi(x)$ be a smooth periodic function in $\mathbb{T}^n$, Δ, ∇, $\nabla\cdot$ be the standard Laplacian, gradient and divergence operators.

(i): Fokker-Planck operator: $\mathcal{F}u = -\Delta u - \nabla \cdot (u \nabla \phi)$

(ii): Witten Laplacian operator: $\mathcal{W}u = -\Delta u + \nabla\phi \cdot \nabla u$
(iii): Schrödinger operator: $\mathcal{S}u = -\Delta u + \left(\frac{1}{4}|\nabla\phi|^2 - \frac{1}{2}\Delta\phi\right) u$

Show that

(a): (5 points) The Fokker-Planck operator can be recast as $\mathcal{F}u = -\nabla \cdot (e^{-\phi}\nabla(e^{\phi}u)$.
(b): (10 points) These three operators have same eigenvalues.
(c): (5 points) Find all equilibrium solutions for these three operators.

3. (15 points) Let $f(x)$ defined on $[0,\ 1]$ be a smooth function with sufficiently many derivatives. $x_i = ih$, where $h = \frac{1}{N}$ and $i = 0, 1, \cdots, N$ are uniformly distributed points in $[0,\ 1]$. What is the highest integer k such that the numerical integration formula

$$I_N = \frac{1}{N}\left(a_0(f(x_0) + f(x_N)) + a_1(f(x_1) + f(x_{N-1})) + \sum_{i=2}^{N-2} f(x_i)\right)$$

is k-th order accurate, namely

$$\left| I_N - \int_0^1 f(x)dx \right| \le Ch^k$$

for a constant C independent of h? Please describe the procedure to obtain the two constants a_0 and a_1 for this k.

4. The wave guide problem is defined as

$$u_t + u_x = 0, \qquad v_t - v_x = 0$$

with the boundary condition

$$u(-1,t) = v(-1,t), \qquad v(1,t) = u(1,t)$$

and the initial condition

$$u(x,0) = f(x), \qquad v(x,0) = g(x).$$

The upwind scheme for the guide problem is defined as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0, \qquad j = -N+1, \cdots, N;$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} - \frac{v_{j+1}^n - v_j^n}{\Delta x} = 0, \qquad j = -N, \cdots, N-1;$$

with the boundary condition

$$u_{-N}^{n+1} = v_{-N}^{n+1}, \qquad v_N^{n+1} = u_N^{n+1}$$

where u_j^n and v_j^n approximate $u(x_j, t^n)$ and $v(x_j, t^n)$ respectively at the grid point (x_j, t^n), with $x_j = j\Delta x$, $t^n = n\Delta t$, $\Delta x = \frac{1}{N}$.

(1) (5 points) For the solution to the wave guide problem with the above boundary condition, prove the energy conservation

$$\frac{d}{dt}\int_{-1}^{1}(u^2+v^2)dx = 0.$$

(2) (5 points) For the numerical solution of the the upwind scheme, if we define the discrete energy as

$$E^n = \sum_{j=-N+1}^{N}(u_j^n)^2 + \sum_{j=-N}^{N-1}(v_j^n)^2,$$

prove the discrete energy stability

$$E^{n+1} \le E^n$$

under a suitable time step restriction $\frac{\Delta t}{\Delta x} \le \lambda_0$. You should first find λ_0.

(3) (10 points) Under the same time step restriction, is the numerical solution stable in the maximum norm? That is, can you prove

$$\max_{-N\le j\le N}\max(|u_j^{n+1}|,|v_j^{n+1}|) \le \max_{-N\le j\le N}\max(|u_j^n|,|v_j^n|)?$$

5. (15 points) Let $G=(V,E)$ be a graph of order n. Let X_1, X_2, $\ldots$, X_q with $2\le q\le \kappa(X)$ be subsets of the vertex set V such that $X = X_1\cup X_2\cup\ldots\cup X_q$. If for each i, $i=1,2,\ldots,q$, and for any pair of nonadjacent vertices $x,y\in X_i$, we have

$$d(x)+d(y)\ge n,$$

then X is cyclable in G (*i.e.*, there is a cycle containing all vertices of X.).

Where $d(x)$ is the degree of x and $\kappa(X)$ is the smallest number of vertices separating two vertices of X if X does not induce a complete subgraph of G, otherwise we put $\kappa(X)=|X|-1$ if $|X|\ge 2$ and $\kappa(X)=1$ if $|X|=1$.

6. (15 points) Let $(F_n)_n$ be the Fibonacci sequence. Namely, $F_0 = 0, F_1=1,\ldots,F_{n+2}=F_{n+1}+F_n$.

Establish a relation between $\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^n$ and F_n and use it to design an efficient algorithm that for a given n computes the n-th Fibonacci number F_n. In particular, it must be *more efficient* than computing F_n in n consecutive steps.

Give an estimate on the number of steps of your algorithm.

Hint: Not that if m is even then

$$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^m = \left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^{m/2}\right)^2$$

and if m is odd then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and $m-1$ is even.

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Probability and Statistics Problems

Team

Please solve 5 out of the following 6 problems.

Problem 1. The characteristic function f of a probability distribution function F is defined by

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x).$$

Show that $f_1(t) = (\cos t)^2$ is a characteristic function and $f_2(t) = |\cos t|$ is not a characteristic function.

Problem 2. Let $I = [0, 1]$ be the unit interval and $\mathcal{B}$ the σ-algebra of Borel sets on I. Let P be the Lebesgue measure on I. Show that on the probability space $(I, \mathcal{B}, P)$ the set of points of x with the following property has probability 1: for all but finitely many rational numbers $p/q \in (0, 1)$,

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{(q \log q)^2}.$$

Problem 3. Let X be an integrable random variable, $\mathcal{G}$ a $\sigma-$algebra, and $Y = E[X|\mathcal{G}]$. Assume that X and Y have the same distribution.

(1) Prove that if X is square-integrable, then $X = Y$, a.s. (i.e. X must be $\mathcal{G}$ measurable) ;

(2) Using a) to prove that for any pair of real numbers a, b with $a < b$, we have $\min\{\max\{X, a\}, b\} = \min\{\max\{Y, a\}, b\}$, and consequently, $X = Y$, a.s.

Problem 4. Let $X_1, \cdots, X_n$ be iid $N(\theta, \sigma^2)$, σ^2 known, and let θ have a double exponential distribution, that is, $\pi(\theta) = e^{-|\theta|/a}/(2a)$, a known. A Bayesian test of the hypothesis $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$ will decide in favor of H_1 if its posterior probability is large.

(a) For a given constant K, calculate the posterior probability that $\theta > K$, that is, $P(\theta > K \mid x_1, \cdots, x_n, a)$.

(b) Find an expression for $\lim_{a\to\infty} P(\theta > K \mid x_1, \cdots, x_n, a)$.

(c) Compare your answer in part (b) to the p-value associated with the classical hypothesis test.

Problem 5. Two sets of interesting ideas emerging in the 1990's are the proposal of model selection with L^1-penalty (e.g., lasso) and the proposal of soft thresholding in simultaneous inferences. Consider a linear model

$$Y = X\beta + \varepsilon,$$

where the set up is as usual (i.e., X is a non-random n by p matrix with $1 < p < n$, $\varepsilon \sim N(0, \sigma^2 \cdot I_n)$ with I_n being the identity matrix). The lasso procedure is to obtain an estimate of the parameter vector β through minimizing

$$(L): \qquad \frac{1}{2}||Y - X\beta||_2^2 + \lambda \cdot ||\beta||_1,$$

which we denote by β_λ^*; here $\lambda > 0$ is a tuning parameter, $||\cdot||_2$ denote the usual L^2 vector norm, and $||\cdot||_1$ denotes the usual L^1 vector norm.

Denote the ordinary least square estimate of β by $\hat{\beta}$, we have

$$\frac{1}{2}||Y - X\beta||_2^2 + \lambda \cdot ||\beta||_1 = \frac{1}{2}||Y - X\hat{\beta}||_2^2 + \frac{1}{2}||X(\beta - \hat{\beta})||_2^2 + \lambda \cdot |\beta||_1. \tag{0.1}$$

Furthermore, if X has orthonormal columns, e.g.,

$$X'X = I_p,$$

then it can be shown that

$$\beta_{\lambda,i}^* = \begin{cases} \hat{\beta}_i - \gamma, & \hat{\beta}_i > \gamma, \\ 0, & |\hat{\beta}_i| \le \gamma, \\ \hat{\beta}_i + \gamma, & \beta_i < -\gamma. \end{cases} \tag{0.2}$$

(0.2) is called the *soft thresholding* of $\hat{\beta}_i$'s. This says that with orthonormal design, lasso solution is equivalent to applying soft thresholding to the ordinary least square solution.

(a) Prove equation (0.1) without assuming X is orthogonal.

(b) Show that the lasso estimator is obtained by (0.2) under the assumption that X is orthogonal, and find the relationship between λ and γ.

Problem 6. Consider a usual linear model $Y = X\beta + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2 \cdot I_n)$ and X has n rows and p columns where $1 < p < n$. Consider a p-dimensional column vector $a \neq 0$.

(a) Show that, if $Xa = 0$, then $a'\beta$ is not estimable.

(b) Prove or disprove that, if $a'\beta$ is not estimable, then $Xa = 0$.

(c) Show that X is full rank if and only if $a'\beta$ are estimable for all a.