Solutions to the 74th William Lowell Putnam Mathematical Competition
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Kiran Kedlaya and Lenny Ng

A1 Suppose otherwise. Then each vertex v is a vertex for five faces, all of which have different labels, and so the sum of the labels of the five faces incident to v is at least $0+1+2+3+4=10$. Adding this sum over all vertices v gives $3\times 39=117$, since each face's label is counted three times. Since there are 12 vertices, we conclude that $10\times 12\leq 117$, contradiction.

Remark: One can also obtain the desired result by showing that any collection of five faces must contain two faces that share a vertex; it then follows that each label can appear at most 4 times, and so the sum of all labels is at least $4(0+1+2+3+4)=40>39$, contradiction.

A2 Suppose to the contrary that $f(n)=f(m)$ with $n<m$, and let $n\cdot a_1\cdots a_r$, $m\cdot b_1\cdots b_s$ be perfect squares where $n<a_1<\cdots<a_r$, $m<b_1<\cdots<b_s$, a_r, b_s are minimal and $a_r=b_s$. Then $(n\cdot a_1\cdots a_r)\cdot(m\cdot b_1\cdots b_s)$ is also a perfect square. Now eliminate any factor in this product that appears twice (i.e., if $a_i=b_j$ for some i,j, then delete a_i and b_j from this product). The product of what remains must also be a perfect square, but this is now a product of distinct integers, the smallest of which is n and the largest of which is strictly smaller than $a_r=b_s$. This contradicts the minimality of a_r.

Remark: Sequences whose product is a perfect square occur naturally in the *quadratic sieve* algorithm for factoring large integers. However, the behavior of the function $f(n)$ seems to be somewhat erratic. A trivial upper bound is $f(n)\leq 2n+2$, arising from the sequence $n, n+1, 2n, 2n+2$; a trivial lower bound is $f(n)\geq n+p$ where p is the least prime factor of n.

A3 Suppose on the contrary that $a_0+a_1y+\cdots+a_ny^n$ is nonzero for $0<y<1$. By the intermediate value theorem, this is only possible if $a_0+a_1y+\cdots+a_ny^n$ has the same sign for $0<y<1$; without loss of generality, we may assume that $a_0+a_1y+\cdots+a_ny^n>0$ for $0<y<1$. For the given value of x, we then have

$$a_0x^m+a_1x^{2m}+\cdots+a_nx^{(n+1)m}\geq 0$$

for $m=0,1,\dots$, with strict inequality for $m>0$. Taking the sum over all m is absolutely convergent and hence valid; this yields

$$\frac{a_0}{1-x}+\frac{a_1}{1-x^2}+\cdots+\frac{a_n}{1-x^{n+1}}>0,$$

a contradiction.

A4 Let $w'_1,\dots,w'_k$ be arcs such that: w'_j has the same length as w_j; w'_1 is the same as w_1; and w'_{j+1} is adjacent to w'_j (i.e., the last digit of w'_j comes right before the first digit of w'_{j+1}). Since w_j has length $Z(w_j)+N(w_j)$, the sum of the lengths of $w_1,\dots,w_k$ is $k(Z+N)$, and so the concatenation of $w'_1,\dots,w'_k$ is a string of $k(Z+N)$ consecutive digits around the circle. (This string may wrap around the circle, in which case some of these digits may appear more than once in the string.) Break this string into k arcs $w''_1,\dots,w''_k$ each of length $Z+N$, each adjacent to the previous one. (Note that if the number of digits around the circle is m, then $Z+N\leq m$ since $Z(w_j)+N(w_j)\leq m$ for all j, and thus each of $w''_1,\dots,w''_k$ is indeed an arc.)

We claim that for some $j=1,\dots,k$, $Z(w''_j)=Z$ and $N(w''_j)=N$ (where the second equation follows from the first since $Z(w''_j)+N(w''_j)=Z+N$). Otherwise, since all of the $Z(w''_j)$ differ by at most 1, either $Z(w''_j)\leq Z-1$ for all j or $Z(w''_j)\geq Z+1$ for all j. In either case, $|kZ-\sum_j Z(w'_j)|=|kZ-\sum_j Z(w''_j)|\geq k$. But since $w_1=w'_1$, we have $|kZ-\sum_j Z(w'_j)|=|\sum_{j=1}^k(Z(w_j)-Z(w'_j))|=|\sum_{j=2}^k(Z(w_j)-Z(w'_j))|\leq \sum_{j=2}^k|Z(w_j)-Z(w'_j)|\leq k-1$, contradiction.

A5 Let $A_1,\dots,A_m$ be points in $\mathbb{R}^3$, and let $\hat{n}_{ijk}$ denote a unit vector normal to $\Delta A_iA_jA_k$ (unless A_i, A_j, A_k are collinear, there are two possible choices for $\hat{n}_{ijk}$). If $\hat{n}$ is a unit vector in $\mathbb{R}^3$, and $\Pi_{\hat{n}}$ is a plane perpendicular to $\hat{n}$, then the area of the orthogonal projection of $\Delta A_iA_jA_k$ onto $\Pi_{\hat{n}}$ is $\mathrm{Area}(\Delta A_iA_jA_k)|\hat{n}_{ijk}\cdot\hat{n}|$. Thus if $\{a_{ijk}\}$ is area definite for $\mathbb{R}^2$, then for any $\hat{n}$,

$$\sum a_{ijk}\mathrm{Area}(\Delta A_iA_jA_k)|\hat{n}_{ijk}\cdot\hat{n}|\geq 0.$$

Note that integrating $|\hat{n}_{ijk}\cdot\hat{n}|$ over $\hat{n}\in S^2$, the unit sphere in $\mathbb{R}^3$, with respect to the natural measure on S^2 gives a positive number c, which is independent of $\hat{n}_{ijk}$ since the measure on S^2 is rotation-independent. Thus integrating the above inequality over $\hat{n}$ gives $c\sum a_{ijk}\mathrm{Area}(\Delta A_iA_jA_k)\geq 0$. It follows that $\{a_{ijk}\}$ is area definite for $\mathbb{R}^3$, as desired.

Remark: It is not hard to check (e.g., by integration in spherical coordinates) that the constant c occurring above is equal to 2π. It follows that for any convex body C in $\mathbb{R}^3$, the average over $\hat{n}$ of the area of the projection of C onto $\Pi_{\hat{n}}$ equals $1/4$ of the surface area of C.

More generally, let C be a convex body in $\mathbb{R}^n$. For $\hat{n}$ a unit vector, let $\Pi_{\hat{n}}$ denote the hyperplane through the origin perpendicular to $\hat{n}$. Then the average over $\hat{n}$ of the volume of the projection of C onto $\Pi_{\hat{n}}$ equals a constant (depending only on n) times the $(n-1)$-dimensional surface area of C.

Statements of this form inhabit the field of *inverse problems*, in which one attempts to reconstruct information about a geometric object from low-dimensional samples. This field has important applications in imaging and tomography.

A6 (by Harm Derksen) Consider the generating functions

$$f(x,y) = \sum_{(a,b)\in S} x^a y^b,$$
$$g(x,y) = \sum_{(a,b)\in \mathbb{Z}^2} w(a,b) x^a y^b.$$

Then $A(S)$ is the constant coefficient of the Laurent polynomial $h(x,y) = f(x,y)f(x^{-1},y^{-1})g(x,y)$. We may compute this coefficient by averaging over unit circles:

$$\begin{aligned}(2\pi)^2 A(S) &= \int_0^{2\pi}\int_0^{2\pi} h(e^{is},e^{it})\,dt\,ds \\ &= \int_0^{2\pi}\int_0^{2\pi} \left|f(e^{is},e^{it})\right|^2 g(e^{is},e^{it})\,dt\,ds.\end{aligned}$$

Consequently, it is enough to check that $g(e^{is},e^{it})$ is a nonnegative real number for all $s,t\in\mathbb{R}$. But $g(e^{is},e^{it}) = 16G(\cos s,\cos t)$ for

$$G(z,w) = zw + z^2 + w^2 - z^2w - zw^2 - z^2w^2.$$

If $z,w \in [-1,1]$ and $zw \geq 0$, then

$$G(z,w) = zw(1-zw) + z^2(1-w) + w^2(1-z) \geq 0.$$

If $z,w \in [-1,1]$ and $zw \leq 0$, then

$$G(z,w) = (z+w)^2 - zw(1+z)(1+w) \geq 0.$$

Hence $g(e^{is},e^{it}) \geq 0$ as desired.

B1 Note that

$$\begin{aligned}c(2k+1)c(2k+3) &= (-1)^k c(k)(-1)^{k+1}c(k+1) \\ &= -c(k)c(k+1) \\ &= -c(2k)c(2k+2).\end{aligned}$$

It follows that $\sum_{n=2}^{2013} c(n)c(n+2) = \sum_{k=1}^{1006}(c(2k)c(2k+2) + c(2k+1)c(2k+3)) = 0$, and so the desired sum is $c(1)c(3) = -1$.

B2 We claim that the maximum value of $f(0)$ is 3. This is attained for $N=2$, $a_1 = \frac{4}{3}$, $a_2 = \frac{2}{3}$: in this case $f(x) = 1 + \frac{4}{3}\cos(2\pi x) + \frac{2}{3}\cos(4\pi x) = 1 + \frac{4}{3}\cos(2\pi x) + \frac{2}{3}(2\cos^2(2\pi x) - 1) = \frac{1}{3}(2\cos(2\pi x)+1)^2$ is always nonnegative.

Now suppose that $f = 1 + \sum_{n=1}^N a_n \cos(2\pi n x) \in C$. When n is an integer, $\cos(2\pi n/3)$ equals 0 if $3|n$ and $-1/2$ otherwise. Thus $a_n\cos(2\pi n/3) = -a_n/2$ for all n, and $f(1/3) = 1 - \sum_{n=1}^N (a_n/2)$. Since $f(1/3) \geq 0$, $\sum_{n=1}^N a_n \leq 2$, whence $f(0) = 1 + \sum_{n=1}^N a_n \leq 3$.

B3 Yes, such numbers must exist. To define them, we make the following observations.

Lemma 1. *For any $i \in \{1,\dots,n\}$, if there exists any $S \in P$ containing i, then there exist $S,T \in P$ such that S is the disjoint union of T with $\{i\}$.*

Proof. Let S be an element of P containing i of minimum cardinality. By (ii), there must be a subset $T \subset S$ containing P with exactly one fewer element than S. These sets have the desired form. □

Lemma 2. *Suppose $S_1,S_2,T_1,T_2 \in P$ have the property that for some $i \in \{1,\dots,n\}$, S_1 is the disjoint union of T_1 with $\{i\}$ and S_2 is the disjoint union of T_2 with $\{i\}$. Then*

$$f(S_1) - f(T_1) = f(S_2) - f(T_2).$$

Proof. By (i) we have

$$\begin{aligned}f(T_1\cup T_2\cup\{i\}) &= f(S_1) + f(T_2) - f(T_1\cap T_2) \\ f(T_1\cup T_2\cup\{i\}) &= f(T_1) + f(S_2) - f(T_1\cap T_2),\end{aligned}$$

from which the claim follows immediately. □

We now define $f_1,\dots,f_n$ as follows. If i does not appear in any element of P, we put $f_i = 0$. Otherwise, by Lemma 1, we can find $S,T \in P$ such that S is the disjoint union of T with $\{i\}$. We then set $f_i = f(S) - f(T)$; by Lemma 2, this does not depend on the choice of S,T.

To check that $f(S) = \sum_{i\in S} f_i$ for $S \in P$, note first that $\emptyset \in P$ by repeated application of (ii) and that $f(\emptyset) = 0$ by hypothesis. This provides the base case for an induction on the cardinality of S; for any nonempty $S \in P$, we may apply (ii) to find $T \subset S$ such that S is the disjoint union of T and some singleton set $\{j\}$. By construction and the induction hypothesis, we have $f(S) = f(T) + f_j = j + \sum_{i\in T} f_i = \sum_{i\in S} f_i$ as desired.

B4 Write $f_0(x) = f(x) - \mu(f)$ and $g_0(x) = g(x) - \mu(g)$, so that $\int_0^1 f_0(x)^2\,dx = \mathrm{Var}(f)$, $\int_0^1 g_0(x)^2\,dx = \mathrm{Var}(g)$, and $\int_0^1 f_0(x)\,dx = \int_0^1 g_0(x)\,dx = 0$. Now since $|g(x)| \leq M(g)$ for all x, $0 \leq \int_0^1 f_0(x)^2(M(g)^2 - g(x)^2)\,dx = \mathrm{Var}(f)M(g)^2 - \int_0^1 f_0(x)^2 g(x)^2\,dx$, and similarly $0 \leq \mathrm{Var}(g)M(f)^2 - \int_0^1 f(x)^2 g_0(x)^2\,dx$. Summing gives

$$\mathrm{Var}(f)M(g)^2 + \mathrm{Var}(g)M(f)^2 \geq \int_0^1 (f_0(x)^2 g(x)^2 + f(x)^2 g_0(x)^2)\,dx. \tag{1}$$

Now

$$\begin{aligned}&\int_0^1 (f_0(x)^2 g(x)^2 + f(x)^2 g_0(x)^2)\,dx - \mathrm{Var}(fg) \\ &= \int_0^1 \left(f_0(x)^2 g(x)^2 + f(x)^2 g_0(x)^2 - \left(f(x)g(x) - \int_0^1 f(y)g(y)\,dy\right)^2\right) dx;\end{aligned}$$

substituting $f_0(x) + \mu(f)$ for $f(x)$ everywhere and $g_0(x) + \mu(g)$ for $g(x)$ everywhere, and using the fact that $\int_0^1 f_0(x)\,dx = \int_0^1 g_0(x)\,dx = 0$, we can expand and simplify the right hand side of this equation to obtain

$$\begin{aligned}
&\int_0^1 (f_0(x)^2 g(x)^2 + f(x)^2 g_0(x)^2)\,dx - \mathrm{Var}(fg) \\
&= \int_0^1 f_0(x)^2 g_0(x)^2\,dx \\
&- 2\mu(f)\mu(g)\int_0^1 f_0(x)g_0(x)\,dx + (\int_0^1 f_0(x)g_0(x)\,dx)^2 \\
&\geq -2\mu(f)\mu(g)\int_0^1 f_0(x)g_0(x)\,dx.
\end{aligned}$$

Because of (1), it thus suffices to show that

$$2\mu(f)\mu(g)\int_0^1 f_0(x)g_0(x)\,dx \leq \mathrm{Var}(f)M(g)^2 + \mathrm{Var}(g)M(f)^2. \tag{2}$$

Now since $(\mu(g)f_0(x) - \mu(f)g_0(x))^2 \geq 0$ for all x, we have

$$\begin{aligned}
2\mu(f)\mu(g)\int_0^1 f_0(x)g_0(x)\,dx &\leq \int_0^1 (\mu(g)^2 f_0(x)^2 + \mu(f)^2 g_0(x)^2)dx \\
&= \mathrm{Var}(f)\mu(g)^2 + \mathrm{Var}(g)\mu(f)^2 \\
&\leq \mathrm{Var}(f)M(g)^2 + \mathrm{Var}(g)M(f)^2,
\end{aligned}$$

establishing (2) and completing the proof.

B5 **First solution:** We assume $n \geq 1$ unless otherwise specified. For T a set and S_1, S_2 two subsets of T, we say that a function $f : T \to T$ *iterates* S_1 *into* S_2 if for each $x \in S_1$, there is a $j \geq 0$ such that $f^{(j)}(x) \in S_2$.

Lemma 1. *Fix $k \in X$. Let $f, g : X \to X$ be two functions such that f iterates X into $\{1,\dots,k\}$ and $f(x) = g(x)$ for $x \in \{k+1,\dots,n\}$. Then g also iterates X into $\{1,\dots,k\}$.*

Proof. For $x \in X$, by hypothesis there exists a nonnegative integer j such that $f^{(j)}(x) \in \{1,\dots,k\}$. Choose the integer j as small as possible; then $f^{(i)}(x) \in \{k+1,\dots,n\}$ for $0 \leq i < j$. By induction on i, we have $f^{(i)}(x) = g^{(i)}(x)$ for $i = 0,\dots,j$, so in particular $g^{(j)}(x) \in \{1,\dots,k\}$. This proves the claim. □

We proceed by induction on $n-k$, the case $n-k=0$ being trivial. For the induction step, we need only confirm that the number x of functions $f : X \to X$ which iterate X into $\{1,\dots,k+1\}$ but not into $\{1,\dots,k\}$ is equal to n^{n-1}. These are precisely the functions for which there is a unique cycle C containing only numbers in $\{k+1,\dots,n\}$ and said cycle contains $k+1$. Suppose C has length $\ell \in \{1,\dots,n-k\}$. For a fixed choice of ℓ, we may choose the underlying set of C in $\binom{n-k-1}{\ell-1}$ ways and the cycle structure in $(\ell-1)!$ ways. Given C, the functions f we want are the ones that act on C as specified and iterate X into $\{1,\dots,k\} \cup C$. By Lemma 1, the number of such functions is $n^{-\ell}$ times the total number of functions that iterate X into $\{1,\dots,k\} \cup C$. By the induction hypothesis, we compute the number of functions which iterate X into $\{1,\dots,k+1\}$ but not into $\{1,\dots,k\}$ to be

$$\sum_{\ell=1}^{n-k} (n-k-1)\cdots(n-k-\ell+1)(k+\ell)n^{n-\ell-1}$$

By rewriting this as a telescoping sum, we get

$$\begin{aligned}
&\sum_{\ell=1}^{n-k} (n-k-1)\cdots(n-k-\ell+1)(n)n^{n-\ell-1} \\
&- \sum_{\ell=1}^{n-k} (n-k-1)\cdots(n-k-\ell+1)(n-k-\ell)n^{n-\ell-1} \\
&= \sum_{\ell=0}^{n-k-1} (n-k-1)\cdots(n-k-\ell)n^{n-\ell-1} \\
&- \sum_{\ell=1}^{n-k} (n-k-1)\cdots(n-k-\ell)n^{n-\ell-1} \\
&= n^{n-1}.
\end{aligned}$$

as desired.

Second solution: For T a set, $f : T \to T$ a function, and S a subset of T, we define the *contraction* of f at S as the function $g : \{*\} \cup (T-S) \to \{*\} \cup (T-S)$ given by

$$g(x) = \begin{cases} * & x = * \\ * & x \neq *, f(x) \in S \\ f(x) & x \neq *, f(x) \notin S. \end{cases}$$

Lemma 2. *For $S \subseteq X$ of cardinality $\ell \geq 0$, there are $\ell n^{n-\ell-1}$ functions $f : \{*\} \cup X \to \{*\} \cup X$ with $f^{-1}(*) = \{*\} \cup S$ which iterate X into $\{*\}$.*

Proof. We induct on n. If $\ell = n$ then there is nothing to check. Otherwise, put $T = f^{-1}(S)$, which must be nonempty. The contraction g of f at $\{*\} \cup S$ is then a function on $\{*\} \cup (X-S)$ with $f^{-1}(*) = \{*\} \cup T$ which iterates $X - S$ into $\{*\}$. Moreover, for given T, each such g arises from $\ell^{\#T}$ functions of the desired form. Summing over T and invoking the induction hypothesis, we see that the number of functions f is

$$\begin{aligned}
&\sum_{k=1}^{n-\ell} \binom{n-\ell}{k} \ell^k \cdot k(n-\ell)^{n-\ell-k-1} \\
&= \sum_{k=1}^{n-\ell} \binom{n-\ell-1}{k-1} \ell^k (n-\ell)^{n-\ell-k} = \ell n^{n-\ell-1}
\end{aligned}$$

as claimed. □

We now count functions $f : X \to X$ which iterate X into $\{1,\dots,k\}$ as follows. By Lemma 1 of the first solution, this count equals n^k times the number of functions with $f(1) = \cdots = f(k) = 1$ which iterate X into $\{1,\dots,k\}$. For such a function f, put $S = \{k+1,\dots,n\} \cap f^{-1}(\{1,\dots,k\})$ and let g be the contraction of f at $\{1,\dots,k\}$; then $g^{-1}(*) = * \cup \{S\}$ and g iterates its domain into $*$. By Lemma 2, for $\ell = \#S$, there are

$\ell(n-k)^{n-k-\ell-1}$ such functions g. For given S, each such g gives rise to k^ℓ functions f with $f(1) = \cdots = f(k) = 1$ which iterate X into $\{1,\dots,k\}$. Thus the number of such functions f is

$$\begin{aligned}
&\sum_{\ell=0}^{n-k} \binom{n-k}{\ell} k^\ell \ell (n-k)^{n-k-\ell-1} \\
&= \sum_{\ell=0}^{n-k} \binom{n-k-1}{\ell-1} k^\ell (n-k)^{n-k-\ell} \\
&= kn^{n-k-1}.
\end{aligned}$$

The desired count is this times n^k, or kn^{n-1} as desired.

Remark: Functions of the sort counted in Lemma 2 can be identified with rooted trees on the vertex set $\{*\} \cup X$ with root $*$. Such trees can be counted using *Cayley's formula*, a special case of *Kirchoff's matrix tree theorem*. The matrix tree theorem can also be used to show directly that the number of rooted forests on n vertices with k fixed roots is kn^{n-k-1}; the desired count follows immediately from this formula plus Lemma 1.

B6 We show that the only winning first move for Alice is to place a stone in the central space. We start with some terminology.

By a *block* of stones, we mean a (possibly empty) sequence of stones occupying consecutive spaces. By the *extremal blocks*, we mean the (possibly empty) maximal blocks adjacent to the left and right ends of the playing area.

We refer to a legal move consisting of placing a stone in an empty space as a move of *type 1*, and any other legal move as being of *type 2*. For $i = 0,\dots,n$, let P_i be the collection of positions containing i stones. Define the *end zone* as the union $Z = P_{n-1} \cup P_n$. In this language, we make the following observations.

- Any move of type 1 from P_i ends in P_{i+1}.
- Any move of type 2 from P_n ends in P_{n-1}.
- For $i < n$, any move of type 2 from P_i ends in $P_i \cup P_{i+1}$.
- At this point, we see that the number of stones cannot decrease until we reach the end zone.
- For $i < n-1$, if we start at a position in P_i where the extremal blocks have length a,b, then the only possible moves to P_i decrease one of a,b while leaving the other unchanged (because they are separated by at least two empty spaces). In particular, no repetition is possible within P_i, so the number of stones must eventually increase to $i+1$.
- From any position in the end zone, the legal moves are precisely to the other positions in the end zone which have not previously occurred. Consequently, after the first move into the end zone, the rest of the game consists of enumerating all positions in the end zone in some order.
- At this point, we may change the rules without affecting the outcome by eliminating the rule on repetitions and declaring that the first player to move into the end zone loses (because $\#Z = n+1$ is even).

To determine who wins in each position, number the spaces of the board $1,\dots,n$ from left to right. Define the *weight* of a position to be the sum of the labels of the occupied spaces, reduced modulo $n+1$. The weight of the initial position is evidently 0, and a move at space s of either type, always adds exactly s to the weight.

We now verify that a position of weight s outside of the end zone is a win for the player to move if and only if $s \neq (n+1)/2$. We check this for positions in P_i for $i = n-2,\dots,0$ by descending induction. For positions in P_{n-2}, the only safe moves are in the extremal blocks; we may thus analyze these positions as two-pile Nim with pile sizes equal to the lengths of the extremal blocks. In particular, a position is a win for the player to move if and only if the extremal blocks are unequal, in which case the winning move is to equalize the blocks. In other words, a position is a win for the player to move unless the empty spaces are at s and $n-s$ for some $s \in \{1,\dots,(n-1)/2\}$, and indeed these are precisely the positions for which the weight equals $(1+\cdots+n) - n \equiv (n+1)/2 \pmod{n+1}$. Given the analysis of positions in P_{i+1} for some i, it is clear that if a position in P_i has weight $s \neq (n+1)/2$, there is a winning move at space t where $s - t \equiv (n+1)/2 \pmod{n}$, whereas if $s = (n+1)/2$ then no move leads to a winning position.

It thus follows that the unique winning move for Alice at her first turn is to move at the central space, as claimed.

Remark: Despite the existence of a simple description of the winning positions, it is nonetheless necessary to go through the preliminary analysis in order to establish the nature of the end zone and to ensure that the repetition clause does not affect the availability of moves outside of the end zone.

Remark: It is easy to see that Alice's winning strategy is to ensure that after each of her moves, the stones are placed symmetrically and the central space is occupied. However, it is somewhat more complicated to describe Bob's winning strategy without the modular interpretation.