International Mathematical Olympiad — 1996

Day 1

July 10, 1997

1. Let $ABCD$ be a rectangular board, with $AB = 20$ and $BC = 12$. The board is divided into 20×12 squares. Let r be a given positive integer. A coin can be moved from one square to another if and only if the distance between the centres of the two squares is $\sqrt{r}$. The task is to find a sequence of moves taking the coin from the square with A as a vertex to the square with B as a vertex.

   (a) Show that the task cannot be done if r is divisible by 2 or 3.

   (b) Prove that the task can be done if $r = 73$.

   (c) Can the task be done when $r = 97$?

2. Let P be a point inside triangle ABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D and E be the incentres of triangles APB and APC respectively. Show that AP, BD, and CF meet at a point.

3. Let $S = \{0,\ 1,\ 2,\ \ldots\}$. Find all functions f defined on S taking their values in S such that $f(m + f(n)) = f(f(m)) + f(n)$ for all m and n in S.

Day 2

July 11, 1996

4. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares.

5. Let $ABCDEF$ be a convex hexagon, such that $AB \| ED$, $BC \| FE$, and $CD \| AF$. Let R_A, R_C, and R_E denote the circumradii of triangles FAB, BCD, and DEF respectively, and let p denote the perimeter of the hexagon. Prove that $R_A + R_C + R_E \geq p/2$.

6. Let n, p, and q be positive integers with $n > p + q$. Let $x_0, x_1, \ldots, x_n$ be integers satisfying the following conditions:

   (a) $x_0 = x_n = 0$, and

   (b) For each integer i, $1 <= i <= n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

   Show that there exists a pair (i, j) of indices with $i < j$, and $(i, j) \neq (0, 0)$, such that $x_i = x_j$.