IMO 2011 shortlist: the final 6

For the International Mathematical Olympiad 2011 the problem selection committee prepared the "shortlist" consisting of 30 problems and answers. The following pages contain the 6 problems that were chosen by the jury as contest problems.

The formulation of the 6 problems given here is the formulation from the shortlist. The wording of the actual contest problems is slightly different (see `www.imo-official.com`).

The remaining 24 problems on the shortlist will be released for public viewing after the IMO 2012.

The problem selection committee consisted of

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Problem 1 (Mexico)

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1 + a_2 + a_3 + a_4$, let p_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

Problem 2 (United Kingdom)

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

Problem 3 (Belarus)

Let f be a function from the set of real numbers to itself that satisfies

$$f(x + y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Problem 4 (Iran)

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

Problem 5 (Iran)

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

Problem 6 (Japan)

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

Problem 1 (Mexico)

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1 + a_2 + a_3 + a_4$, let p_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

Answer. The sets A for which p_A is maximal are the sets the form $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer. For all these sets p_A is 4.

Solution. Firstly, we will prove that the maximum value of p_A is at most 4. Without loss of generality, we may assume that $a_1 < a_2 < a_3 < a_4$. We observe that for each pair of indices (i, j) with $1 \le i < j \le 4$, the sum $a_i + a_j$ divides s_A if and only if $a_i + a_j$ divides $s_A - (a_i + a_j) = a_k + a_l$, where k and l are the other two indices. Since there are 6 distinct pairs, we have to prove that at least two of them do not satisfy the previous condition. We claim that two such pairs are (a_2, a_4) and (a_3, a_4). Indeed, note that $a_2 + a_4 > a_1 + a_3$ and $a_3 + a_4 > a_1 + a_2$. Hence $a_2 + a_4$ and $a_3 + a_4$ do not divide s_A. This proves $p_A \le 4$.

Now suppose $p_A = 4$. By the previous argument we have

$$\begin{array}{lcl} a_1 + a_4 \mid a_2 + a_3 & \text{and} & a_2 + a_3 \mid a_1 + a_4, \\ a_1 + a_2 \mid a_3 + a_4 & \text{and} & a_3 + a_4 \nmid a_1 + a_2, \\ a_1 + a_3 \mid a_2 + a_4 & \text{and} & a_2 + a_4 \nmid a_1 + a_3. \end{array}$$

Hence, there exist positive integers m and n with $m > n \ge 2$ such that

$$\begin{cases} a_1 + a_4 = a_2 + a_3 \\ m(a_1 + a_2) = a_3 + a_4 \\ n(a_1 + a_3) = a_2 + a_4. \end{cases}$$

Adding up the first equation and the third one, we get $n(a_1 + a_3) = 2a_2 + a_3 - a_1$. If $n \ge 3$, then $n(a_1 + a_3) > 3a_3 > 2a_2 + a_3 > 2a_2 + a_3 - a_1$. This is a contradiction. Therefore $n = 2$. If we multiply by 2 the sum of the first equation and the third one, we obtain

$$6a_1 + 2a_3 = 4a_2,$$

while the sum of the first one and the second one is

$$(m + 1)a_1 + (m - 1)a_2 = 2a_3.$$

Adding up the last two equations we get

$$(m + 7)a_1 = (5 - m)a_2.$$

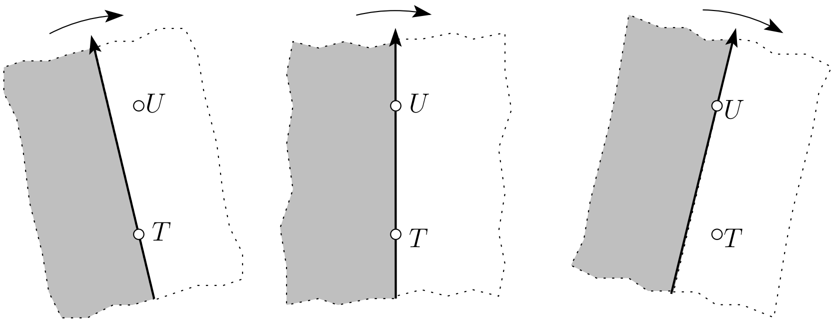
It follows that $5 - m \geq 1$, because the left-hand side of the last equation and a_2 are positive. Since we have $m > n = 2$, the integer m can be equal only to either 3 or 4. Substituting $(3, 2)$ and $(4, 2)$ for (m, n) and solving the previous system of equations, we find the families of solutions $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer.

Problem 2 (United Kingdom)

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

Solution. Give the rotating line an orientation and distinguish its sides as the *oranje side* and the *blue side*. Notice that whenever the pivot changes from some point T to another point U, after the change, T is on the same side as U was before. Therefore, the number of elements of $\mathcal{S}$ on the oranje side and the number of those on the blue side remain the same throughout the whole process (except for those moments when the line contains two points).



First consider the case that $|\mathcal{S}| = 2n + 1$ is odd. We claim that through any point $T \in \mathcal{S}$, there is a line that has n points on each side. To see this, choose an oriented line through T containing no other point of $\mathcal{S}$ and suppose that it has $n + r$ points on its oranje side. If $r = 0$ then we have established the claim, so we may assume that $r \neq 0$. As the line rotates through $180°$ around T, the number of points of $\mathcal{S}$ on its oranje side changes by 1 whenever the line passes through a point; after $180°$, the number of points on the oranje side is $n - r$. Therefore there is an intermediate stage at which the oranje side, and thus also the blue side, contains n points.

Now select the point P arbitrarily, and choose a line through P that has n points of $\mathcal{S}$ on each side to be the initial state of the windmill. We will show that during a rotation over $180°$, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and select a line ℓ through T that separates $\mathcal{S}$ into equal halves. The point T is the unique point of $\mathcal{S}$ through which a line in this direction can separate the points of $\mathcal{S}$ into equal halves (parallel translation would disturb the balance). Therefore, when the windmill line is parallel to ℓ, it must be ℓ itself, and so pass through T.

Next suppose that $|\mathcal{S}| = 2n$. Similarly to the odd case, for every $T \in \mathcal{S}$ there is an oriented line through T with $n-1$ points on its oranje side and n points on its blue side. Select such an oriented line through an arbitrary P to be the initial state of the windmill.

We will now show that during a rotation over $360°$, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and an oriented line ℓ through T that separates $\mathcal{S}$ into two subsets with $n-1$ points on its oranje and n points on its blue side. Again, parallel translation would change the numbers of points on the two sides, so when the windmill line is parallel to ℓ with the same orientation, the windmill line must pass through T.

Comment. One may shorten this solution in the following way.

Suppose that $|\mathcal{S}| = 2n+1$. Consider any line ℓ that separates $\mathcal{S}$ into equal halves; this line is unique given its direction and contains some point $T \in \mathcal{S}$. Consider the windmill starting from this line. When the line has made a rotation of $180°$, it returns to the same location but the oranje side becomes blue and vice versa. So, for each point there should have been a moment when it appeared as pivot, as this is the only way for a point to pass from on side to the other.

Now suppose that $|\mathcal{S}| = 2n$. Consider a line having $n-1$ and n points on the two sides; it contains some point T. Consider the windmill starting from this line. After having made a rotation of $180°$, the windmill line contains some different point R, and each point different from T and R has changed the color of its side. So, the windmill should have passed through all the points.

Problem 3 (Belarus)

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x)) \tag{1}$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Solution 1. Substituting $y = t - x$, we rewrite (1) as

$$f(t) \leq tf(x) - xf(x) + f(f(x)). \tag{2}$$

Consider now some real numbers a, b and use (2) with $t = f(a)$, $x = b$ as well as with $t = f(b)$, $x = a$. We get

$$f(f(a)) - f(f(b)) \leq f(a)f(b) - bf(b),$$
$$f(f(b)) - f(f(a)) \leq f(a)f(b) - af(a).$$

Adding these two inequalities yields

$$2f(a)f(b) \geq af(a) + bf(b).$$

Now, substitute $b = 2f(a)$ to obtain $2f(a)f(b) \geq af(a) + 2f(a)f(b)$, or $af(a) \leq 0$. So, we get

$$f(a) \geq 0 \quad \text{for all } a < 0. \tag{3}$$

Now suppose $f(x) > 0$ for some real number x. From (2) we immediately get that for every $t < \dfrac{xf(x) - f(f(x))}{f(x)}$ we have $f(t) < 0$. This contradicts (3); therefore

$$f(x) \leq 0 \quad \text{for all real } x, \tag{4}$$

and by (3) again we get $f(x) = 0$ for all $x < 0$.

We are left to find $f(0)$. Setting $t = x < 0$ in (2) we get

$$0 \leq 0 - 0 + f(0),$$

so $f(0) \geq 0$. Combining this with (4) we obtain $f(0) = 0$.

Solution 2. We will also use the condition of the problem in form (2). For clarity we divide the argument into four steps.

Step 1. We begin by proving that f attains nonpositive values only. Assume that there exist some real number z with $f(z) > 0$. Substituting $x = z$ into (2) and setting $A = f(z)$, $B = -zf(z) - f(f(z))$ we get $f(t) \leq At + B$ for all real t. Hence, if for any positive real number t we substitute $x = -t$, $y = t$ into (1), we get

$$\begin{aligned} f(0) &\leq tf(-t) + f(f(-t)) \leq t(-At + B) + Af(-t) + B \\ &\leq -t(At - B) + A(-At + B) + B = -At^2 - (A^2 - B)t + (A + 1)B. \end{aligned}$$

But surely this cannot be true if we take t to be large enough. This contradiction proves that we have indeed $f(x) \leq 0$ for all real numbers x. Note that for this reason (1) entails

$$f(x + y) \leq yf(x) \tag{5}$$

for all real numbers x and y.

Step 2. We proceed by proving that f has at least one zero. If $f(0) = 0$, we are done. Otherwise, in view of Step 1 we get $f(0) < 0$. Observe that (5) tells us now $f(y) \leq yf(0)$ for all real numbers y. Thus we can specify a positive real number a that is so large that $f(a)^2 > -f(0)$. Put $b = f(a)$ and substitute $x = b$ and $y = -b$ into (5); we learn $-b^2 < f(0) \leq -bf(b)$, i.e. $b < f(b)$. Now we apply (2) to $x = b$ and $t = f(b)$, which yields

$$f(f(b)) \leq \big(f(b) - b\big)f(b) + f(f(b)),$$

i.e. $f(b) \geq 0$. So in view of Step 1, b is a zero of f.

Step 3. Next we show that if $f(a) = 0$ and $b < a$, then $f(b) = 0$ as well. To see this, we just substitute $x = b$ and $y = a - b$ into (5), thus getting $f(b) \geq 0$, which suffices by Step 1.

Step 4. By Step 3, the solution of the problem is reduced to showing $f(0) = 0$. Pick any zero r of f and substitute $x = r$ and $y = -1$ into (1). Because of $f(r) = f(r - 1) = 0$ this gives $f(0) \geq 0$ and hence $f(0) = 0$ by Step 1 again.

Comment 1. Both of these solutions also show $f(x) \leq 0$ for all real numbers x. As one can see from Solution 1, this task gets much easier if one already knows that f takes nonnegative values for sufficiently small arguments. Another way of arriving at this statement, suggested by the proposer, is as follows:

Put $a = f(0)$ and substitute $x = 0$ into (1). This gives $f(y) \leq ay + f(a)$ for all real numbers y. Thus if for any real number x we plug $y = a - x$ into (1), we obtain

$$f(a) \leq (a - x)f(x) + f(f(x)) \leq (a - x)f(x) + af(x) + f(a)$$

and hence $0 \leq (2a - x)f(x)$. In particular, if $x < 2a$, then $f(x) \geq 0$.

Having reached this point, one may proceed almost exactly as in the first solution to deduce $f(x) \leq 0$ for all x. Afterwards the problem can be solved in a few lines as shown in steps 3 and 4 of the second solution.

Comment 2. The original problem also contained the question whether a nonzero function satisfying the problem condition exists. Here we present a family of such functions.

Notice first that if $g : (0, \infty) \longrightarrow [0, \infty)$ denotes any function such that

$$g(x + y) \geq yg(x) \tag{6}$$

for all positive real numbers x and y, then the function f given by

$$f(x) = \begin{cases} -g(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{7}$$

automatically satisfies (1). Indeed, we have $f(x) \leq 0$ and hence also $f(f(x)) = 0$ for all real numbers x. So (1) reduces to (5); moreover, this inequality is nontrivial only if x and y are positive. In this last case it is provided by (6).

Now it is not hard to come up with a nonzero function g obeying (6). E.g. $g(z) = Ce^z$ (where C is a positive constant) fits since the inequality $e^y > y$ holds for all (positive) real numbers y. One may also consider the function $g(z) = e^z - 1$; in this case, we even have that f is continuous.

Problem 4 (Iran)

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

Answer. The number $f(n)$ of ways of placing the n weights is equal to the product of all odd positive integers less than or equal to $2n-1$, i.e. $f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$.

Solution 1. Assume $n \geq 2$. We claim

$$f(n) = (2n-1)f(n-1). \tag{1}$$

Firstly, note that after the first move the left pan is always at least 1 heavier than the right one. Hence, any valid way of placing the n weights on the scale gives rise, by not considering weight 1, to a valid way of placing the weights $2, 2^2, \ldots, 2^{n-1}$.

If we divide the weight of each weight by 2, the answer does not change. So these $n-1$ weights can be placed on the scale in $f(n-1)$ valid ways. Now we look at weight 1. If it is put on the scale in the first move, then it has to be placed on the left side, otherwise it can be placed either on the left or on the right side, because after the first move the difference between the weights on the left pan and the weights on the right pan is at least 2. Hence, there are exactly $2n-1$ different ways of inserting weight 1 in each of the $f(n-1)$ valid sequences for the $n-1$ weights in order to get a valid sequence for the n weights. This proves the claim.

Since $f(1) = 1$, by induction we obtain for all positive integers n

$$f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

Comment 1. The word "compute" in the statement of the problem is probably too vague. An alternative but more artificial question might ask for the smallest n for which the number of valid ways is divisible by 2011. In this case the answer would be 1006.

Comment 2. It is useful to remark that the answer is the same for any set of weights where each weight is heavier than the sum of the lighter ones. Indeed, in such cases the given condition is equivalent to asking that during the process the heaviest weight on the balance is always on the left pan.

Comment 3. Instead of considering the lightest weight, one may also consider the last weight put on the balance. If this weight is 2^{n-1} then it should be put on the left pan. Otherwise it may be put on any pan; the inequality would not be violated since at this moment the heaviest weight is already put onto the left pan. In view of the previous comment, in each of these $2n-1$ cases the number of ways to place the previous weights is exactly $f(n-1)$, which yields (1).

Solution 2. We present a different way of obtaining (1). Set $f(0) = 1$. Firstly, we find a recurrent formula for $f(n)$.

Assume $n \geq 1$. Suppose that weight 2^{n-1} is placed on the balance in the i-th move with $1 \leq i \leq n$. This weight has to be put on the left pan. For the previous moves we have $\binom{n-1}{i-1}$ choices of the weights and from Comment 2 there are $f(i-1)$ valid ways of placing them on the balance. For later moves there is no restriction on the way in which the weights are to be put on the pans. Therefore, all $(n-i)!2^{n-i}$ ways are possible. This gives

$$f(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} f(i-1)(n-i)!2^{n-i} = \sum_{i=1}^{n} \frac{(n-1)!f(i-1)2^{n-i}}{(i-1)!}. \tag{2}$$

Now we are ready to prove (1). Using $n-1$ instead of n in (2) we get

$$f(n-1) = \sum_{i=1}^{n-1} \frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}.$$

Hence, again from (2) we get

$$\begin{aligned} f(n) &= 2(n-1)\sum_{i=1}^{n-1} \frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!} + f(n-1) \\ &= (2n-2)f(n-1) + f(n-1) = (2n-1)f(n-1), \end{aligned}$$

QED.

Comment. There exist different ways of obtaining the formula (2). Here we show one of them.

Suppose that in the first move we use weight 2^{n-i+1}. Then the lighter $n-i$ weights may be put on the balance at any moment and on either pan. This gives $2^{n-i} \cdot (n-1)!/(i-1)!$ choices for the moves (moments and choices of pan) with the lighter weights. The remaining $i-1$ moves give a valid sequence for the $i-1$ heavier weights and this is the only requirement for these moves, so there are $f(i-1)$ such sequences. Summing over all $i = 1, 2, \ldots, n$ we again come to (2).

Problem 5 (Iran)

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

Solution 1. Suppose that x and y are two integers with $f(x) < f(y)$. We will show that $f(x) \mid f(y)$. By taking $m = x$ and $n = y$ we see that

$$f(x-y) \mid |f(x) - f(y)| = f(y) - f(x) > 0,$$

so $f(x-y) \leq f(y) - f(x) < f(y)$. Hence the number $d = f(x) - f(x-y)$ satisfies

$$-f(y) < -f(x-y) < d < f(x) < f(y).$$

Taking $m = x$ and $n = x - y$ we see that $f(y) \mid d$, so we deduce $d = 0$, or in other words $f(x) = f(x-y)$. Taking $m = x$ and $n = y$ we see that $f(x) = f(x-y) \mid f(x) - f(y)$, which implies $f(x) \mid f(y)$.

Solution 2. We split the solution into a sequence of claims; in each claim, the letters m and n denote arbitrary integers.

Claim 1. $f(n) \mid f(mn)$.

Proof. Since trivially $f(n) \mid f(1 \cdot n)$ and $f(n) \mid f((k+1)n) - f(kn)$ for all integers k, this is easily seen by using induction on m in both directions. □

Claim 2. $f(n) \mid f(0)$ and $f(n) = f(-n)$.

Proof. The first part follows by plugging $m = 0$ into Claim 1. Using Claim 1 twice with $m = -1$, we get $f(n) \mid f(-n) \mid f(n)$, from which the second part follows. □

From Claim 1, we get $f(1) \mid f(n)$ for all integers n, so $f(1)$ is the minimal value attained by f. Next, from Claim 2, the function f can attain only a finite number of values since all these values divide $f(0)$.

Now we prove the statement of the problem by induction on the number N_f of values attained by f. In the base case $N_f \leq 2$, we either have $f(0) \neq f(1)$, in which case these two numbers are the only values attained by f and the statement is clear, or we have $f(0) = f(1)$, in which case we have $f(1) \mid f(n) \mid f(0)$ for all integers n, so f is constant and the statement is obvious again.

For the induction step, assume that $N_f \geq 3$, and let a be the least positive integer with $f(a) > f(1)$. Note that such a number exists due to the symmetry of f obtained in Claim 2.

Claim 3. $f(n) \neq f(1)$ if and only if $a \mid n$.

Proof. Since $f(1) = \cdots = f(a-1) < f(a)$, the claim follows from the fact that

$$f(n) = f(1) \iff f(n+a) = f(1).$$

So it suffices to prove this fact.

Assume that $f(n) = f(1)$. Then $f(n+a) \mid f(a) - f(-n) = f(a) - f(n) > 0$, so $f(n+a) \leq f(a) - f(n) < f(a)$; in particular the difference $f(n+a) - f(n)$ is stricly smaller than $f(a)$. Furthermore, this difference is divisible by $f(a)$ and nonnegative since $f(n) = f(1)$ is the least value attained by f. So we have $f(n+a) - f(n) = 0$, as desired. For the converse direction we only need to remark that $f(n+a) = f(1)$ entails $f(-n-a) = f(1)$, and hence $f(n) = f(-n) = f(1)$ by the forward implication. $\square$

We return to the induction step. So let us take two arbitrary integers m and n with $f(m) \leq f(n)$. If $a \nmid m$, then we have $f(m) = f(1) \mid f(n)$. On the other hand, suppose that $a \mid m$; then by Claim 3 $a \mid n$ as well. Now define the function $g(x) = f(ax)$. Clearly, g satisfies the conditions of the problem, but $N_g < N_f - 1$, since g does not attain $f(1)$. Hence, by the induction hypothesis, $f(m) = g(m/a) \mid g(n/a) = f(n)$, as desired.

Comment. After the fact that f attains a finite number of values has been established, there are several ways of finishing the solution. For instance, let $f(0) = b_1 > b_2 > \cdots > b_k$ be all these values. One may show (essentially in the same way as in Claim 3) that the set $S_i = \{n : f(n) \geq b_i\}$ consists exactly of all numbers divisible by some integer $a_i \geq 0$. One obviously has $a_i \mid a_{i-1}$, which implies $f(a_i) \mid f(a_{i-1})$ by Claim 1. So, $b_k \mid b_{k-1} \mid \cdots \mid b_1$, thus proving the problem statement.

Moreover, now it is easy to describe all functions satisfying the conditions of the problem. Namely, all these functions can be constructed as follows. Consider a sequence of nonnegative integers $a_1, a_2, \ldots, a_k$ and another sequence of positive integers $b_1, b_2, \ldots, b_k$ such that $|a_k| = 1$, $a_i \neq a_j$ and $b_i \neq b_j$ for all $1 \leq i < j \leq k$, and $a_i \mid a_{i-1}$ and $b_i \mid b_{i-1}$ for all $i = 2, \ldots, k$. Then one may introduce the function

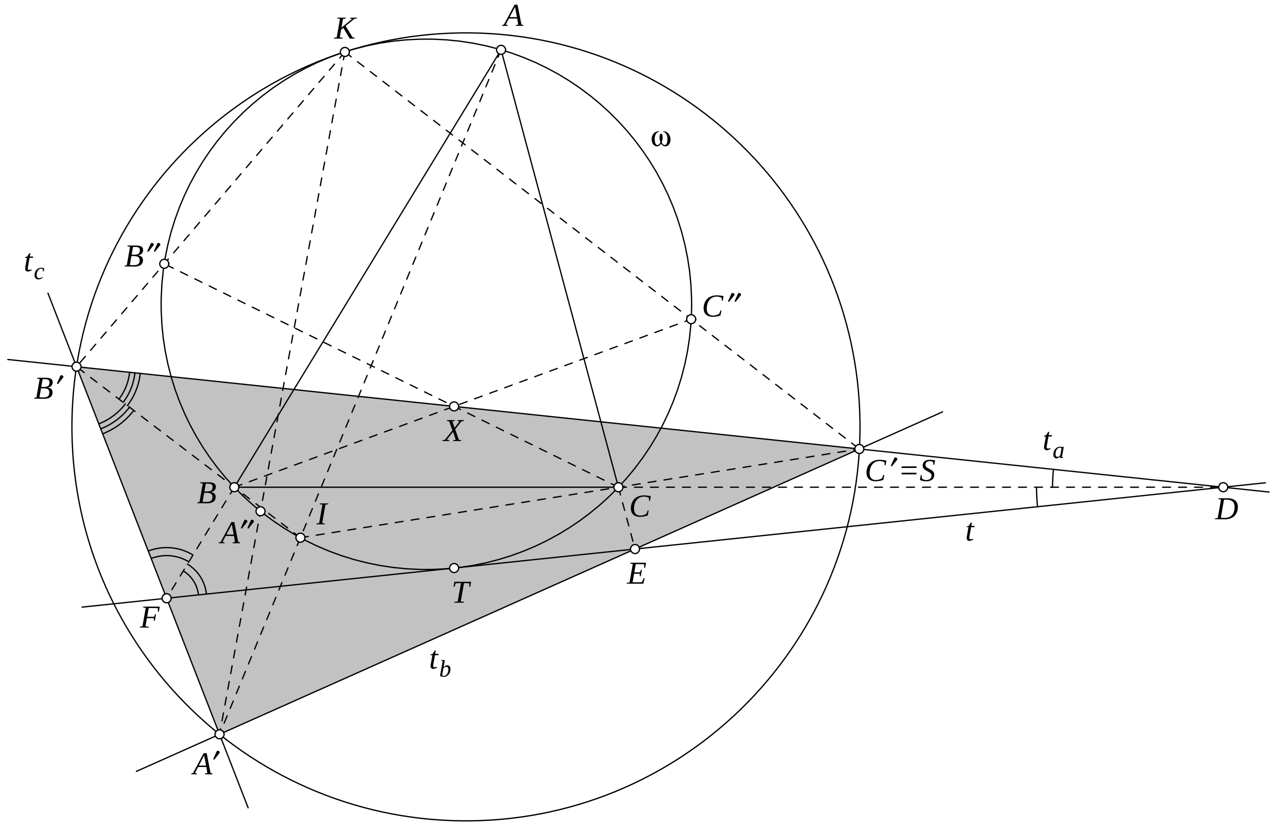
$$f(n) = b_{i(n)}, \qquad \text{where} \quad i(n) = \min\{i : a_i \mid n\}.$$

These are all the functions which satisfy the conditions of the problem.

Problem 6 (Japan)

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

To avoid a large case distinction, we will use the notion of *oriented angles*. Namely, for two lines ℓ and m, we denote by $\angle(\ell, m)$ the angle by which one may rotate ℓ anticlockwise to obtain a line parallel to m. Thus, all oriented angles are considered modulo 180°.



Solution 1. Denote by T the point of tangency of t and ω. Let $A' = t_b \cap t_c$, $B' = t_a \cap t_c$, $C' = t_a \cap t_b$. Introduce the point A'' on ω such that $TA = AA''$ ($A'' \neq T$ unless TA is a diameter). Define the points B'' and C'' in a similar way.

Since the points C and B are the midpoints of arcs TC'' and TB'', respectively, we have

$$\begin{aligned}\angle(t, B''C'') &= \angle(t, TC'') + \angle(TC'', B''C'') = 2\angle(t, TC) + 2\angle(TC'', BC'') \\ &= 2\big(\angle(t, TC) + \angle(TC, BC)\big) = 2\angle(t, BC) = \angle(t, t_a).\end{aligned}$$

It follows that t_a and $B''C''$ are parallel. Similarly, $t_b \parallel A''C''$ and $t_c \parallel A''B''$. Thus, either the triangles $A'B'C'$ and $A''B''C''$ are homothetic, or they are translates of each other. Now we will prove that they are in fact homothetic, and that the center K of the homothety belongs to ω. It would then follow that their circumcircles are also homothetic with respect to K and are therefore tangent at this point, as desired.

We need the two following claims.

Claim 1. The point of intersection X of the lines $B''C$ and BC'' lies on t_a.

Proof. Actually, the points X and T are symmetric about the line BC, since the lines CT and CB'' are symmetric about this line, as are the lines BT and BC''. $\square$

Claim 2. The point of intersection I of the lines BB' and CC' lies on the circle ω.

Proof. We consider the case that t is not parallel to the sides of ABC; the other cases may be regarded as limit cases. Let $D = t \cap BC$, $E = t \cap AC$, and $F = t \cap AB$.

Due to symmetry, the line DB is one of the angle bisectors of the lines $B'D$ and FD; analogously, the line FB is one of the angle bisectors of the lines $B'F$ and DF. So B is either the incenter or one of the excenters of the triangle $B'DF$. In any case we have $\angle(BD, DF) + \angle(DF, FB) + \angle(B'B, B'D) = 90°$, so

$$\angle(B'B, B'C') = \angle(B'B, B'D) = 90° - \angle(BC, DF) - \angle(DF, BA) = 90° - \angle(BC, AB).$$

Analogously, we get $\angle(C'C, B'C') = 90° - \angle(BC, AC)$. Hence,

$$\angle(BI, CI) = \angle(B'B, B'C') + \angle(B'C', C'C) = \angle(BC, AC) - \angle(BC, AB) = \angle(AB, AC),$$

which means exactly that the points A, B, I, C are concyclic. $\square$

Now we can complete the proof. Let K be the second intersection point of $B'B''$ and ω. Applying PASCAL's theorem to hexagon $KB''CIBC''$ we get that the points $B' = KB'' \cap IB$ and $X = B''C \cap BC''$ are collinear with the intersection point S of CI and $C''K$. So $S = CI \cap B'X = C'$, and the points C', C'', K are collinear. Thus K is the intersection point of $B'B''$ and $C'C''$ which implies that K is the center of the homothety mapping $A'B'C'$ to $A''B''C''$, and it belongs to ω.

Solution 2. Define the points T, A', B', and C' in the same way as in the previous solution. Let X, Y, and Z be the symmetric images of T about the lines BC, CA, and AB, respectively. Note that the projections of T on these lines form a SIMSON line of T with respect to ABC, therefore the points X, Y, Z are also collinear. Moreover, we have $X \in B'C'$, $Y \in C'A'$, $Z \in A'B'$.

Denote $\alpha = \angle(t, TC) = \angle(BT, BC)$. Using the symmetry in the lines AC and BC, we get

$$\angle(BC, BX) = \angle(BT, BC) = \alpha \quad \text{and} \quad \angle(XC, XC') = \angle(t, TC) = \angle(YC, YC') = \alpha.$$

Since $\angle(XC, XC') = \angle(YC, YC')$, the points X, Y, C, C' lie on some circle ω_c. Define the circles ω_a and ω_b analogously. Let ω' be the circumcircle of triangle $A'B'C'$.

Now, applying Miquel's theorem to the four lines $A'B'$, $A'C'$, $B'C'$, and XY, we obtain that the circles ω', ω_a, ω_b, ω_c intersect at some point K. We will show that K lies on ω, and that the tangent lines to ω and ω' at this point coincide; this implies the problem statement.

Due to symmetry, we have $XB = TB = ZB$, so the point B is the midpoint of one of the arcs XZ of circle ω_b. Therefore $\angle(KB, KX) = \angle(XZ, XB)$. Analogously, $\angle(KX, KC) = \angle(XC, XY)$. Adding these equalities and using the symmetry in the line BC we get

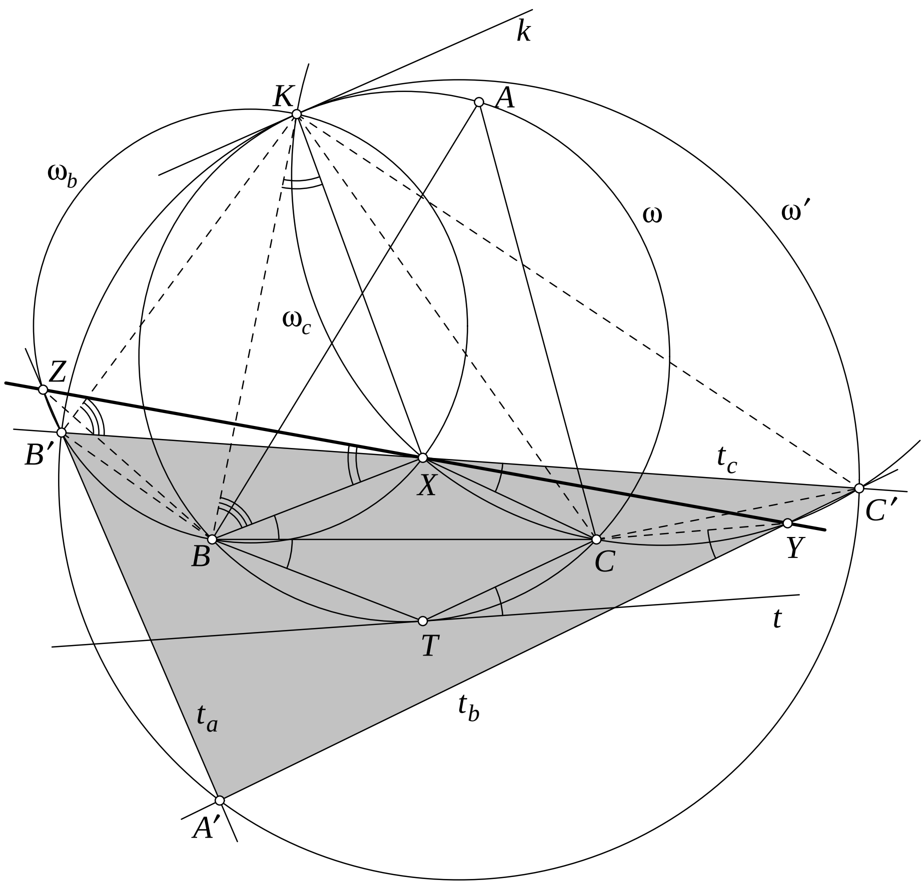
$$\angle(KB, KC) = \angle(XZ, XB) + \angle(XC, XZ) = \angle(XC, XB) = \angle(TB, TC).$$

Therefore, K lies on ω.

Next, let k be the tangent line to ω at K. We have

$$\begin{aligned}\angle(k, KC') &= \angle(k, KC) + \angle(KC, KC') = \angle(KB, BC) + \angle(XC, XC')\\ &= \big(\angle(KB, BX) - \angle(BC, BX)\big) + \alpha = \angle(KB', B'X) - \alpha + \alpha = \angle(KB', B'C'),\end{aligned}$$

which means exactly that k is tangent to ω'.



Comment. There exist various solutions combining the ideas from the two solutions presented above. For instance, one may define the point X as the reflection of T with respect to the line BC, and then introduce the point K as the second intersection point of the circumcircles of $BB'X$ and $CC'X$. Using the fact that BB' and CC' are the bisectors of $\angle(A'B', B'C')$ and $\angle(A'C', B'C')$ one can show successively that $K \in \omega$, $K \in \omega'$, and that the tangents to ω and ω' at K coincide.