Problem 1. Let $G$ be a finite subgroup of $\text{GL}(V)$ where $V$ is an $n$-dimensional complex vector space.

(a) (5 points) Let $H = \{h \in G : hv = \eta(h)v \text{ for some } \eta(h) \in \mathbb{C}^\times \text{ and all } v \in V\}$. Prove that $H$ is a normal subgroup of $G$ and that the map $h \mapsto \eta(h)$ is an isomorphism between $H$ and its image in $\mathbb{C}^\times$.

(b) (5 points) Let $\chi_V$ be the character function of $G$ acting on $V$, i.e., $\chi_V(g) = \text{tr}(g)$ with $g$ viewed as an automorphism of $V$. Prove $|\chi_V(g)| \leq n$ for all $g \in G$, and the equality holds if and only if $g \in H$.

(c) (10 points) Let $W$ be an irreducible representation of $G$. Then $W$ is isomorphic to a direct summand of $V^{\otimes m}$ for some $m$ (as representations of $G$).

Problem 2. Let $a_1, \ldots, a_n$ be nonnegative real numbers.

(a) (6 points) Prove that the $n \times n$ matrix $A = (t^{a_i + a_j})$ is positive semi-definite for every real number $t > 0$. Find the rank of $A$.

(b) (7 points) Let $B = (c_{ij})_{n \times n}$ be an $n \times n$-matrix with $c_{ij} = \frac{1}{1 + a_i + a_j}$. Prove that $A$ is a positive semi-definite matrix.

(c) (7 points) Prove that $B$ is positive definite if and only if $a_i$ are all distinct.

Problem 3. Consider the equations $X^2 - 82Y^2 = \pm 2$

(a) (5 points) Show that if $(x, y)$ is a solution for $X^2 - 82Y^2 = \pm 2$, then $(9x - 82y, x - 9y)$ is a solution for $X^2 - 82Y^2 = \mp 2$.

(b) (7 points) Show that the equations have solutions over $\mathbb{Z}/p^n\mathbb{Z}$ for any $n$ and odd prime $p$.

(c) (8 points) Show that the equations have no solutions over $\mathbb{Z}$.

Problem 4. Let $S$ and $T$ be nonabelian finite simple groups, and write $G = S \times T$.

(a) (7 points) Show that the total number of normal subgroups of $G$ is four.

(b) (6 points) If $S$ and $T$ are isomorphic, show that $G$ has a maximal proper subgroup not containing either direct factor.

(c) (7 points) If $G$ has a maximal proper subgroup that contains neither of the direct factors of $G$, show that $S$ and $T$ are isomorphic.
Problem 5. (20 points) Let $\mathbb{F}$ be a finite field and $f_i \in \mathbb{F}[X_1, X_2, ..., X_n]$ be polynomials of degree $d_i$, where $1 \leq i \leq r$, such that $f_i(0, ..., 0) = 0$ for all $i$. Show that if

$$n > \sum_{i=1}^{r} d_i,$$

then there exists nonzero solution to the system of equations: $f_i = 0$, for all $1 \leq i \leq r$. (Hint: you may first verify that the number of integral solutions is congruent to the following number modulo $p$

$$\sum_{X \in \mathbb{F}^n} \prod_{i=1}^{r} (1 - f_i(X)^{q-1}).$$

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Problem 6.

(a) (5 points) Let $A$ and $B$ be two real $n \times n$ matrices such that $AB = BA$. Show that $\det(A^2 + B^2) \geq 0$.

(b) (15 points) Generalize this to the case of $k$ pairwise commuting matrices.