S.-T. Yau College Student Mathematics Contests 2014

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies

$$\sup_{x,y\in\mathbb{R}} |f(x+y) - f(x) - f(y)| < \infty.$$

If we have $\lim_{n\to\infty, n\in\mathbb{N}} \dfrac{f(n)}{n} = 2014$, prove $\sup_{x\in\mathbb{R}} |f(x) - 2014\, x| < \infty$.

2. Let $f_1, ..., f_n$ are analytic functions on $D = \{z||z| < 1\}$ and continuous on $\bar{D}$, prove that $\phi(z) = |f_1(z)| + |f_2(z)| + ... + |f_n(z)|$ achieves maximum values at the boundary ∂D.

3. Prove that if there is a conformal mapping between the annulus $\{z|r_1 < |z| < r_2\}$ and the annulus $\{z|\rho_1 < |z| < \rho_2\}$, then $\frac{r_2}{r_1} = \frac{\rho_2}{\rho_1}$.

4. Let $U(\xi)$ be a bounded function on $\mathbb{R}$ with finitely many points of discontinuity, prove that

$$P_U(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-\xi)^2 + y^2} U(\xi) d\xi$$

is a harmonic function on the upper half plane $\{z \in \mathbb{C}|Imz > 0\}$ and it converges to $U(\xi)$ as $z \to \xi$ at a point ξ where $U(\xi)$ is continuous.

5. Let $f \in L^2(\mathbb{R})$ and let $\hat{f}$ be its Fourier transform. Prove that

$$\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi \geq \frac{(\int_{-\infty}^{\infty} |f(x)|^2 dx)^2}{16\pi^2},$$

under the condition that the two integrals on the left are bounded.

(Hint: Operators $f(x) \to xf(x)$ and $\hat{f}(\xi) \to \xi\hat{f}(\xi)$ after Fourier transform are non-commuting operators. The inequality is a version of the uncertainty principle.)

6. Let Ω be an open domain in the complex plane $\mathbb{C}$. Let $\mathbb{H}$ be the subspace of $L^2(\Omega)$ consisting of holomorphic functions on Ω.

a) Show that $\mathbb{H}$ is a closed subspace of $L^2(\Omega)$, and hence is a Hilbert space with inner product

$$(f, g) = \int_{\Omega} f(z)\bar{g}(z) dxdy, \text{ where } z = x + iy.$$

b) If $\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathbb{H}$, then

$$\Sigma_{n=0}^{\infty}|\phi_n(z)|^2 \leq \frac{c^2}{d(z,\Omega^c)}, \text{ for } z \in \Omega.$$

c) The sum

$$B(z,w) = \Sigma_{n=0}^{\infty}\phi_n(z)\bar{\phi}_n(w)$$

converges absolutely for $(z,w) \in \Omega \times \Omega$, and is independent of the choice of the orthonormal basis.