S.-T. Yau College Student Mathematics Contests 2014

Applied Math. and Computational Math.

Individual

Please solve as many problems as you can!

1. (20 pts) Ming Antu (1692-1763) is one of the greatest Chinese/Mongolian mathematicians. In the 1730s, he first established and used what was later to be known as Catalan numbers (Euler (1707-1763) rediscovered them around 1756; Belgian mathematician Eugene Catalan (1814-1894) "rediscovered" them again in 1838),

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \quad n = 0, 1, 2, \cdots$$

and Ming Antu derived the following half-angle formula in 1730:

$$\sin^2 \frac{\theta}{2} = \sum_{n=1}^{\infty} c_{n-1} \left(\frac{\sin\theta}{2}\right)^{2n}$$

Prove this formula.

Hint: you may use generating function

$$F(z) = \sum_{n=0}^{\infty} c_n z^n$$

and show that $\sum_{m+k=n} c_m c_k = c_{n+1}$ *and then show* $zF(z)^2 = F(z) - 1$.

2. *Many algorithms, including polynomial factorisation in finite fields, require to compute* $\gcd(f(X), X^N - 1)$ *for a polynomial* f *of reasonably small degree* n *and a binomial* $X^N - 1$ *of very large degree* N*. Since* N *is very large the direct application of the Euclid algorithm is very inefficient.*

Questions:

- (i) (10 pts) Suggest a more efficient approach the direct computation of $\gcd(f(X), X^N - 1)$ via the Euclid algorithm.
- (ii) (10 pts) Generalise it to $\gcd(f(X), A_1 X^{N_1} + \ldots + A_m X^{N_m} + A_{m+1})$.

Hint: If for three polynomials f, g *and* h *we have* $g \equiv h \pmod f$ *then*

$$\gcd(f, g) = \gcd(f, h).$$

3. For solving the following partial differential equation

$$u_t + f(u)_x = 0, \qquad 0 \le x \le 1 \tag{1}$$

where $f'(u) \ge 0$, with periodic boundary condition, we can use the following semi-discrete upwind scheme

$$\frac{d}{dt}u_j + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = 0, \qquad j = 1, 2, \cdots, N, \tag{2}$$

with periodic boundary condition

$$u_0 = u_N, \tag{3}$$

where $u_j = u_j(t)$ approximates $u(x_j, t)$ at the grid point $x = x_j = j\Delta x$, with $\Delta x = \frac{1}{N}$.

(i) (15 pts) Prove the following L^2 stability of the scheme

$$\frac{d}{dt}E(t) \le 0 \tag{4}$$

where $E(t) = \sum_{j=1}^{N} |u_j|^2 \Delta x$.

(ii) (15 pts) Do you believe (4) is true for $E(t) = \sum_{j=1}^{N} |u_j|^{2p} \Delta x$ for *arbitrary* integer $p \ge 1$? If yes, prove the result. If not, give a counter example.

4. Let A be an $n \times n$ matrix with real and positive eigenvalues and b be a given vector. Consider the solution of $Ax = b$ by the following Richardson's iteration

$$x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b$$

where ω is a damping coefficient. Let λ_1 and λ_n be the smallest and the largest eigenvalues of A. Let $G_\omega = I - \omega A$.

(i) (4 points) Prove that the Richardson's iteration converges if and only if

$$0 < \omega < \frac{2}{\lambda_n}.$$

(ii) (8 points) Prove that the optimal choice of ω is given by

$$\omega_{\text{opt}} = \frac{2}{\lambda_1 + \lambda_n}.$$

Prove also that

$$\rho(G_\omega) = \begin{cases} 1 - \omega\lambda_1 & \omega \le \omega_{\text{opt}} \\ (\lambda_n - \lambda_1)/(\lambda_n + \lambda_1) & \omega = \omega_{\text{opt}} \\ \omega\lambda_n - 1 & \omega \ge \omega_{\text{opt}} \end{cases}$$

where $\rho(G_\omega)$ is the spectral radius of G_ω.

(iii) (8 points) Prove that, if A is symmetric and positive definite, then

$$\rho(G_{\omega_{\rm opt}}) = \frac{\kappa_2(A) - 1}{\kappa_2(A) + 1}$$

where $\kappa_2(A)$ is the spectral condition number of A.

5. (10 pts) For solving the following heat equation on interval

$$u_t = u_{xx}, \qquad 0 \le x \le 1 \tag{5}$$

with boundary condition

$$u(0) = u_0, \quad u(1) = u_1, \tag{6}$$

we first discretize the interval $[0, 1]$ into N subintervals uniformly, that is, the mesh size $h = 1/N$. We choose a temporal step size k and approximate the solution $u(jh, nk)$ by U_j^n, $j = 1, ..., N-1$, $n = 0, 1, 2, ...$ Using the backward Euler method in time and central finite difference in space, the discrete function U_j^n satisfies:

$$U_j^{n+1} - U_j^n = \lambda(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}), \quad j = 1, ..., N-1, \tag{7}$$

where $\lambda = k/h^2$, and

$$U_0^{n+1} = u_0, \ U_N^{n+1} = u_1.$$

Show that

$$\frac{1}{2}\sum_{j=1}^{N-1}\left((U_j^{n+1})^2 - (U_j^n)^2\right) \le -\lambda \sum_{j=1}^{N-2}(U_{j+1}^{n+1} - U_j^{n+1})^2$$
$$- \frac{\lambda}{2}((U_1^{n+1})^2 + (U_{N-1}^{n+1})^2) + \frac{\lambda}{2}(u_0^2 + u_1^2) \tag{8}$$