S.-T. Yau College Student Mathematics Contests 2014

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_n(X)$. Do the same for S^3 with antipodal points of the equator $S^2 \subset S^3$ identified.

2. Let $M \to \mathbb{R}^3$ be a graph defined by $z = f(u, v)$ where $\{u, v, z\}$ is a Descartes coordinate system in $\mathbb{R}^3$. Suppose that M is a minimal surface. Prove that:
(a) The Gauss curvature K of M can be expressed as

$$K = \Delta \log \left(1 + \frac{1}{W}\right), \quad W := \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2},$$

where Δ denotes the Laplacian with respect to the induce metric on M (i.e., the first fundamental form of M).
(b) If f is defined on the whole uv-plane, then f is a linear function (Bernstein theorem).

3. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

4. Let $p : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. This is a submersion $p : \tilde{M} \to M$ such that for each $x \in \tilde{M}$, $Dp : \ker^{\perp}(Dp) \to T_{p(x)}(M)$ is a linear isometry.

(a) Show that p shortens distances.
(b) If $(\tilde{M}, \tilde{g})$ is complete, so is (M, g).
(c) Show by example that if (M, g) is complete, $(\tilde{M}, \tilde{g})$ may not be complete.

5. Let $\Psi : M \to \mathbb{R}^3$ be an isometric immersion of a compact surface M into $\mathbb{R}^3$. Prove that $\int_M H^2 d\sigma \geq 4\pi$, where H is the mean curvature of M and $d\sigma$ is the area element of M.

6. The unit tangent bundle of S^2 is the subset

$$T^1(S^2) = \{(p, v) \in \mathbb{R}^3 \mid \| p \| = 1,\ (p, v) = 0 \text{ and } \| v \| = 1\}.$$

Show that it is a smooth submanifold of the tangent bundle $T(S^2)$ of S^2 and $T^1(S^2)$ is diffeomorphic to $\mathbb{R}P^3$.