Solutions
Problem 1

Problem 1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Solution. Claim: The sets A for which n_A is maximal are the sets the form $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer. For all these sets n_A is 4.

First we will prove that the maximum value of n_A is at most 4.
Suppose r is a divisor of s_A which is not s_A itself. Then $r \leq \frac{1}{2}s_A$. So, for any positive r_1, r_2 with $r_1 + r_2 = s_A$, either $r_1 = r_2 = \frac{1}{2}s_A$ or at most one of the r_i is a divisor of s_A.
Suppose $n_A \geq 5$. Then the sum of at most one pair does not have to be a divisor of s_A, say (a_2, a_3). The sums $a_1 + a_2$ and $a_3 + a_4$ must both be divisors of s_A, and as their sum is s_A, they must both be $\frac{1}{2}s_A$. The same can be concluded for $a_1 + a_3$ and $a_2 + a_4$. But this means that $a_1 + a_2 = a_1 + a_3$, which implies $a_2 = a_3$, which is not allowed. So $n_A \leq 4$.
Now suppose $n_A = 4$. By the previous argument we can show that the sums of two complementary pairs must equal $\frac{1}{2}s_A$, say (a_1, a_2) and (a_3, a_4). The sums of two other pairs must be some other divisor of s_A, these sums must therefore both be less than $\frac{1}{2}s_A$, and the pairs thus must necessarily share an element, say a_1. This gives us the following set of equations:

$$a_1 + a_2 = \frac{1}{2}s_A \tag{1}$$
$$a_3 + a_4 = \frac{1}{2}s_A \tag{2}$$
$$a_1 + a_3 = \frac{1}{u}s_A \tag{3}$$
$$a_1 + a_4 = \frac{1}{v}s_A \tag{4}$$

Here, u and v are positive integers, with $u, v \geq 3$. They cannot be equal, as that would imply $a_3 = a_4$. Because of symmetry we can assume $u < v$ and thus $\frac{1}{u} > \frac{1}{v}$.
If we add up (3) and (4), we get $2a_1 + a_3 + a_4 = (\frac{1}{u} + \frac{1}{v})s_A$. Substituting (2) yields $2a_1 = (\frac{1}{u} + \frac{1}{v} - \frac{1}{2})s_A$. So, in order for a_1 to be positive, the following must hold:

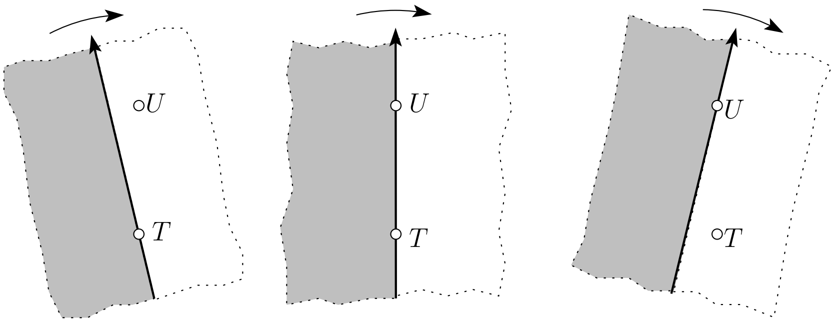
$$\frac{1}{u} + \frac{1}{v} > \frac{1}{2}$$

From $\frac{2}{u} > \frac{1}{u} + \frac{1}{v} > \frac{1}{2}$, it follows that $u < 4$ and hence $u = 3$. But then $\frac{1}{u} + \frac{1}{v} = \frac{1}{3} + \frac{1}{v} > \frac{1}{2}$, which implies $\frac{1}{v} > \frac{1}{6}$ and hence $v < 6$. This leaves $v = 4$ or $v = 5$ as the only options:

- $v = 4$ yields $(a_1, a_2, a_3, a_4) = (\frac{1}{24}s_A, \frac{11}{24}s_A, \frac{7}{24}s_A, \frac{5}{24}s_A)$, implying solutions $\{d, 11d, 7d, 5d\}$, where d is any positive integer.
- $v = 5$ yields $(a_1, a_2, a_3, a_4) = (\frac{1}{60}s_A, \frac{29}{60}s_A, \frac{19}{60}s_A, \frac{11}{60}s_A)$, implying solutions $\{d, 29d, 19d, 11d\}$, where d is any positive integer.

Problem 2. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.
Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Solution. Give the rotating line an orientation and distinguish its sides as the *oranje side* and the *blue side*. Notice that whenever the pivot changes from some point T to another point U, after the change, T is on the same side as U was before. Therefore, the number of elements of $\mathcal{S}$ on the oranje side and the number of those on the blue side remain the same throughout the whole process (except for those moments when the line contains two points).



First consider the case that $|\mathcal{S}| = 2n + 1$ is odd. We claim that through any point $T \in \mathcal{S}$, there is a line that has n points on each side. To see this, choose an oriented line through T containing no other point of $\mathcal{S}$ and suppose that it has $n + r$ points on its oranje side. If $r = 0$ then we have established the claim, so we may assume that $r \neq 0$. As the line rotates through 180° around T, the number of points of $\mathcal{S}$ on its oranje side changes by 1 whenever the line passes through a point; after 180°, the number of points on the oranje side is $n - r$. Therefore there is an intermediate stage at which the oranje side, and thus also the blue side, contains n points.
Now select the point P arbitrarily, and choose a line through P that has n points of $\mathcal{S}$ on each side to be the initial state of the windmill. We will show that during a rotation over 180°, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and select a line ℓ through T that separates $\mathcal{S}$ into equal halves. The point T is the unique point of $\mathcal{S}$ through which a line in this direction can separate the points of $\mathcal{S}$ into equal halves (parallel translation would disturb the balance). Therefore, when the windmill line is parallel to ℓ, it must be ℓ itself, and so pass through T.

Next suppose that $|\mathcal{S}| = 2n$. Similarly to the odd case, for every $T \in \mathcal{S}$ there is an oriented line through T with $n - 1$ points on its oranje side and n points on its blue side. Select such an oriented line through an arbitrary P to be the initial state of the windmill.

We will now show that during a rotation over 360°, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and an oriented line ℓ through T that separates $\mathcal{S}$ into two subsets with $n-1$ points on its oranje and n points on its blue side. Again, parallel translation would change the numbers of points on the two sides, so when the windmill line is parallel to ℓ with the same orientation, the windmill line must pass through T.

Comment. One may shorten this solution in the following way.
Suppose that $|\mathcal{S}| = 2n+1$. Consider any line ℓ that separates $\mathcal{S}$ into equal halves; this line is unique given its direction and contains some point $T \in \mathcal{S}$. Consider the windmill starting from this line. When the line has made a rotation of 180°, it returns to the same location but the oranje side becomes blue and vice versa. So, for each point there should have been a moment when it appeared as pivot, as this is the only way for a point to pass from on side to the other.
Now suppose that $|\mathcal{S}| = 2n$. Consider a line having $n-1$ and n points on the two sides; it contains some point T. Consider the windmill starting from this line. After having made a rotation of 180°, the windmill line contains some different point R, and each point different from T and R has changed the color of its side. So, the windmill should have passed through all the points.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x)) \tag{5}$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Solution. Substituting $y = t - x$, we rewrite (5) as

$$f(t) \leq tf(x) - xf(x) + f(f(x)). \tag{6}$$

Now consider some real numbers a, b and use (6) with $t = f(a)$, $x = b$ as well as with $t = f(b)$, $x = a$. We get

$$f(f(a)) - f(f(b)) \leq f(a)f(b) - bf(b),$$
$$f(f(b)) - f(f(a)) \leq f(a)f(b) - af(a).$$

Adding these two inequalities yields

$$2f(a)f(b) \geq af(a) + bf(b).$$

We substitute $b = 2f(a)$ and obtain $2f(a)f(b) \geq af(a) + 2f(a)f(b)$, or $af(a) \leq 0$. Therefore,

$$f(a) \geq 0 \quad \text{for all } a < 0. \tag{7}$$

Now suppose $f(x) > 0$ for some real number x. From (6), we immediately get that, for every $t < \dfrac{xf(x) - f(f(x))}{f(x)}$, we have $f(t) < 0$. This contradicts (7); therefore

$$f(x) \leq 0 \quad \text{for all real } x, \tag{8}$$

and, by (7) again, we get $f(x) = 0$ for all $x < 0$.
We are left to find $f(0)$. Setting $t = x < 0$ in (6), we obtain $f(x) \leq f(f(x))$. As x is a zero of f, this means

$$0 \leq f(0).$$

Combining this with (8), we obtain $f(0) = 0$.