International Mathematical Olympiad Amsterdam 2011

Problem 4. Let $n > 0$ be an integer. We are given a balance and n weights of weight 2^0, $2^1, \ldots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.
Determine the number of ways in which this can be done.

Solution. Claim: The number $f(n)$ of ways of placing the n weights is equal to the product of all odd positive integers less than or equal to $2n-1$, i.e., $f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$. Other ways of writing the same answer are

$$\frac{(2n)!}{2^n n!} \qquad 1 \cdot 3 \cdot 5 \cdots (2n-1) \qquad \frac{(2n-1)!}{2^{n-1}(n-1)!} \qquad \prod_{k=1}^{n}(2k-1)$$

Recursive Solutions Assume $n \geq 2$. We claim

$$f(n) = (2n-1)f(n-1). \tag{1}$$

Furthermore, $f(1) = 1$ as there is only one way to place the single weight: on the left pan. From this the formula follows by induction.

Easy derivations To begin we note that

$$\sum_{i<k} 2^i < 2^k \tag{2}$$

for every k; this means that at any given moment the largest weight used till then must be on the left pan and that the left pan is always at least 1 unit heavier than the right pan — even 2 units is weight 1 wasn't used yet.
This means that if we take weight 1 out of the sequence we still have a valid sequence of placings but now for weights 2, 2^2, ..., 2^{n-1}. The number of those sequences is $f(n-1)$ and each such sequence gives rise to $2n-1$ sequences for weights 1, 2, 2^2, ..., 2^{n-1} by inserting weight 1: if it goes first it must go left (1 possibility), otherwise it can go either left or right ($2(n-1)$ possibilities, 2 for each of the $n-1$ moments it is placed). It follows that, indeed, $f(n) = (2n-1) \cdot f(n-1)$.
Alternatively one can consider the last weight placed: if it is 2^{n-1} it should go left, otherwise it can go left or right. Therefore we have $2n-1$ possibilities for the last placement. The other weights can be placed in $f(n-1)$ ways, thanks to condition (2); this, again, yields (1).

Slightly harder derivations One can also look at the moment the heaviest weight 2^{n-1} is placed on a pan, necessarily the left pan of course. Say it is placed at the ith move.

The $i-1$ weights that come before can be chosen in $\binom{n-1}{i-1}$ ways and these then yield $f(i-1)$ sequences. The remaining $n-i$ weights can be placed in any order, with an arbitrary choice of pans, that is, in $(n-i)!2^{n-i}$ ways. This gives us the following recursive formula:

$$f(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} f(i-1)(n-i)!2^{n-i} = \sum_{i=1}^{n} \frac{(n-1)!f(i-1)2^{n-i}}{(i-1)!}. \tag{3}$$

In this case we have to set $f(0) = 1$ to seed the recursion.
To derive (1) we substitute $n-1$ for n into (3):

$$f(n-1) = \sum_{i=1}^{n-1} \frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}.$$

Hence, again from (3) we get

$$\begin{aligned} f(n) &= 2(n-1)\sum_{i=1}^{n-1} \frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!} + f(n-1) \\ &= (2n-2)f(n-1) + f(n-1) = (2n-1)f(n-1), \end{aligned}$$

Alternatively we consider which weight is placed first, 2^{i-1} say. Then by the arguments similar to those above we can place weights $2^0, \dots, 2^{i-2}$ in any way whatsoever: we choose $\binom{n-1}{i-1}$ moments and for each such choice we have $2^{i-1}(i-1)!$ placings. The other weights — 2^i, …, 2^{n-1} — Can be placed in $f(n-i)$ valid ways. This gives us

$$f(n) = \sum_{i=1}^{n-1} \binom{n-1}{i-1} 2^{i-1}(i-1)!f(n-i)$$

which can be converted into (3) as above.

Alternative direct solution We create our sequence by imposing a linear order on the weights and at the same time labelling them with L or R.
We order by considering the weights from heaviest to lightest, and determining the number of combinations position/label they can have relative to the heavier weights.
A sequence is valid if and only if each weight that is placed before all heavier weights is placed on the left pan.

- Weight 2^{n-1} is simply put down and its label must be L. This gives one possibility
- If weight 2^{n-2} is put before 2^{n-1} then it *must* receive label L. If is after 2^{n-1} then its label can be L or R. This gives three possibilities
- Weight 2^{n-3} is put in one of the three intervals created by 2^{n-1} and 2^{n-2}; if put in the first interval it gets label L, and it may get an L or R when it is put in one of the other two intervals. This yields five possibilities.
- We continue in this way: weights $2^{n-1}, 2^{n-2}, \dots, 2^{n-i}$ create $i+1$ intervals. These provide $2i+1$ possibilities for the position/label for weight $2^{n-(i+1)}$, namely L if before all and L or R if in the remaining intervals.

In this way we create $1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)$ sequences since we end when $2^{n-(i+1)} = 2^0$, i.e., when $i = n-1$ and we then have $2(n-1)+1 = 2n-1$ possibilities for 2^0.

Problem 5. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

Solution. For reference we write down the given property

$$f(m-n) \mid f(m) - f(n) \tag{$\star$}$$

that holds for f, and all integers m, n.

Suppose that x and y are two integers with $f(x) < f(y)$. We will show that $f(x) \mid f(y)$. By taking $m = x$ and $n = y$ in ($\star$) we see that

$$f(x-y) \mid |f(x) - f(y)| = f(y) - f(x) > 0,$$

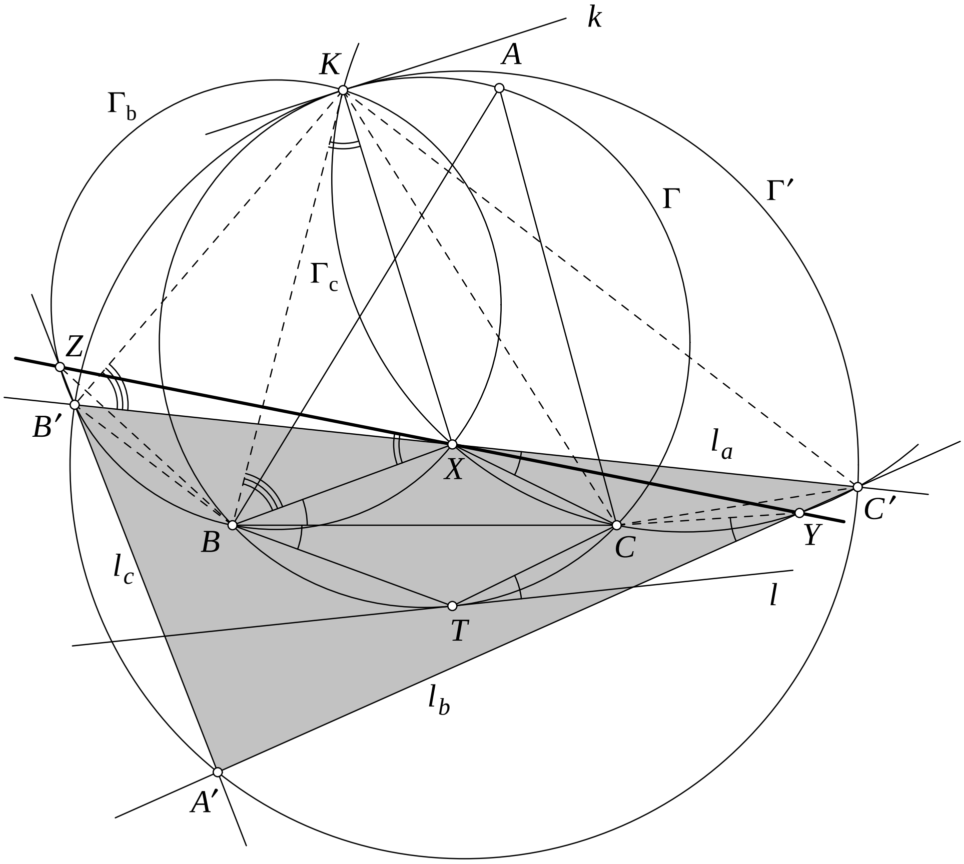
so $f(x-y) \leq f(y) - f(x) < f(y)$. Hence the difference $d = f(x) - f(x-y)$ satisfies

$$-f(y) < -f(x-y) < d < f(x) < f(y).$$

Taking $m = x$ and $n = x - y$ in ($\star$) we see that $f(y) \mid d$, so we deduce $d = 0$, or in other words $f(x) = f(x-y)$. Taking $m = x$ and $n = y$ in ($\star$) we see that $f(x) = f(x-y) \mid f(x) - f(y)$, which implies $f(x) \mid f(y)$.

Problem 6. Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Solution. To avoid a large case distinction, we will use the notion of *oriented angles*. Namely, for two lines m and n, we denote by $\angle(m, n)$ the angle by which one may rotate m anticlockwise to obtain a line parallel to n. Thus, all oriented angles are considered modulo $180°$.



Denote by T the point of tangency of ℓ and Γ. Let $A' = \ell_b \cap \ell_c$, $B' = \ell_a \cap \ell_c$, $C' = \ell_a \cap \ell_b$. Let Γ' be the circumcircle of triangle $A'B'C'$. Let X, Y, and Z be the symmetric images of T about the lines BC, CA, and AB, respectively. These are collinear since they form a *Steiner line* of T with respect to ABC. (We can also deduce this from *Simson's line*.) Moreover, we have $X \in B'C'$, $Y \in C'A'$, $Z \in A'B'$.
Denote $\alpha = \angle(\ell, TC) = \angle(BT, BC)$. Using the symmetry in the lines AC and BC, we get

$$\angle(BC, BX) = \angle(BT, BC) = \alpha \quad \text{and} \quad \angle(XC, XC') = \angle(\ell, TC) = \angle(YC, YC') = \alpha.$$

Since $\angle(XC, XC') = \angle(YC, YC')$, the points X, Y, C, C' lie on some circle Γ_c. Define the circles Γ_a and Γ_b analogously.

Now, applying *Miquel's theorem* to the four lines $A'B'$, $A'C'$, $B'C'$, and XY, we obtain that the circles Γ', Γ_a, Γ_b, Γ_c intersect at some point K. We will show that K lies on Γ, and that the tangent lines to Γ and Γ' at this point coincide; this implies the problem statement.
Due to symmetry, we have $XB = TB = ZB$, so the point B is the midpoint of one of the arcs XZ of circle Γ_b. Therefore $\angle(KB, KX) = \angle(XZ, XB)$. Analogously, $\angle(KX, KC) = \angle(XC, XY)$. Adding these equalities and using the symmetry in the line BC we get

$$\angle(KB, KC) = \angle(XZ, XB) + \angle(XC, XZ) = \angle(XC, XB) = \angle(TB, TC).$$

Therefore, K lies on Γ.
Next, let k be the tangent line to Γ at K. We have

$$\begin{aligned}\angle(k, KC') &= \angle(k, KC) + \angle(KC, KC') = \angle(KB, BC) + \angle(XC, XC') \\ &= \big(\angle(KB, BX) - \angle(BC, BX)\big) + \alpha = \angle(KB', B'X) - \alpha + \alpha = \angle(KB', B'C'),\end{aligned}$$

which means exactly that k is tangent to Γ'.