S.-T. Yau College Student Mathematics Contests 2014

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Calculate the integral:

$$\int_0^\infty \frac{\log x}{1+x^2} dx.$$

2. Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$.

3. Prove that any bounded analytic function F over $\{z|r < |z| < R\}$ can be written as $F(z) = z^\alpha f(z)$, where f is an analytic function over the disk $\{z||z| < R\}$ and α is a constant.

4. Let $D \subset \mathbb{R}^n$ be a bounded open set, $f : \bar{D} \to \bar{D}$ is a smooth map such that its Jacobian $\left|\frac{\partial f}{\partial x}\right| \equiv 1$, where $\bar{D}$ denotes the closure of D. Prove

(a) for each small ball $B_\epsilon(x)$, there exists a positive integer k such that $f^k(B_\epsilon(x)) \cap B_\epsilon(x) \neq \varnothing$, where $B_\epsilon(x)$ denotes the ball centered at x with radius ϵ;

(b) there exists $x \in \bar{D}$ and a sequence $k_1, k_2, \cdots k_j, \cdots$ such that $f^{k_j}(x) \to x$ as $k_j \to \infty$.

5. Let u be a subharmonic function over a domain $\Omega \subset \mathbf{C}$, i.e., it is twice differentiable and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0$. Prove that u achieves its maximum in the interior of Ω only when u is a constant.

6. Suppose that $\phi \in C_0^\infty(\mathbf{R}^n)$, $\int_{\mathbf{R}^n} \phi dx = 1$. Let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon), x \in \mathbf{R}^n, \epsilon > 0$. Prove that if $f \in L^p(\mathbf{R}^n), 1 \leq p < \infty$, then $f * \phi_\epsilon \to f$ in $L^p(\mathbf{R}^n)$, as $\epsilon \to 0$. It is not true for $p = \infty$.

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Probability and Statistics Problems

Team

Please solve the following 5 problems.

Problem 1. Suppose that X_n converges to X in distribution and Y_n converges to a constant c in distribution. Show that

(a) Y_n converges to c in probability;

(b) $X_n Y_n$ converges to cX in distribution.

Problem 2. Let X and Y be two random variables with $|Y| > 0$, a.s.. Let $Z = X/Y$.

(a) Assume the distribution function of (X, Y) has the density $p(x, y)$. What is the density function of Z?

(b) Assume X and Y are independent and X is $N(0, 1)$ distributed, Y has the uniform distribution on $(0, 1)$. Give the density function of Z.

Problem 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, and $\Gamma \in \mathcal{F}$. Prove that the following properties are equivalent:

(i) Γ is independent of $\mathcal{G}$ under P,

(ii) for every probability Q on $(\Omega, \mathcal{F})$, equivalent to P, with dQ/dP being $\mathcal{G}$ measurable, we have $Q(\Gamma) = P(\Gamma)$.

(b) Let X, Y, Z be random variables and Y is integrable. Show that if (X, Y) and Z are independent, then $E[Y|X, Z] = E[Y|X]$.

Problem 4. Let $X_1, ..., X_n$ be i.i.d. $N(0, \sigma^2)$, and let M be the mean of $|X_1|, ..., |X_n|$.

1. Find $c \in R$ so that $\hat{\sigma} = cM$ is a consistent estimator of σ.
2. Determine the limiting distribution for $\sqrt{n}(\hat{\sigma} - \sigma)$.
3. Identify an approximate $(1 - \alpha)\%$ confidence interval for σ.
4. Is $\hat{\sigma} = cM$ asymptotically efficient? Please justify your answer.

Problem 5. The shifted exponential distribution has the density function

$$f(y;\phi,\theta) = 1/\theta \exp\{-(u-\phi)/\theta\}, \qquad y > \phi, \theta > 0.$$

Let $Y_1, \dots, Y_n$ be a random sample from this distribution. Find the maximum likelihood estimator (MLE) of ϕ and θ and the limiting distribution of the MLE.

You may use the following Rényi representation of the order statistics: Let $E_1, \dots, E_n$, be a random sample from the standard exponential distribution (i.e., the above distribution with $\phi = 0, \theta = 1$). Let $E_{(r)}$ denote the r-th order statistics. According to the Rényi representation,

$$E_{(r)} \stackrel{D}{=} \sum_{j=1}^{r} \frac{E_j}{n+1-j}, \qquad r = 1, \dots, n.$$

Here, the symbol $\stackrel{D}{=}$ denotes equal in distribution.

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Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Compute the fundamental and homology groups of the wedge sum of a circle S^1 and a torus $T = S^1 \times S^1$.

2. Given a properly discontinuous action $F : G \times M \to M$ on a smooth manifold M, show that M/G is orientable if and only if M is orientable and $F(g, \cdot)$ preserves the orientation of M. Use this statement to show that the Möbius band is not orientable and that $\mathbb{R}P^n$ is orientable if and only if n is odd.

3. (a) Consider the space Y obtained from $S^2 \times [0, 1]$ by identifying $(x, 0)$ with $(-x, 0)$ and also identifying $(x, 1)$ with $(-x, 1)$, for all $x \in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

(b) Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

4. Prove that a bi-invariant metric on a Lie group G has nonnegative sectional curvature.

5. Let M be the upper half-plane $\mathbb{R}^2_+$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^k}.$$

For which values of k is M complete?

6. Given any nonorientable manifold M show the existence of a smooth orientable manifold $\overline{M}$ which is a double covering of M. Find $\overline{M}$ when M is $\mathbb{R}P^2$ or the Möbius band.

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Algebra and Number Theory

Team

Solve 5 out of 6 problems, or the highest 5 scores will be counted.

Problem 1. Let the special linear group (of order 2)

$$\mathrm{SL}_2(\mathbb{R}) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : \det g = 1\}$$

act on the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ linear fractionally:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

(a) (5 points) Prove that the action is transitive, i.e., for any two $z_1, z_2 \in \mathbb{H}$, there is $g \in \mathrm{SL}_2(\mathbb{R})$ such that $gz_1 = z_2$.

(b) (5 points) For a fixed $z \in \mathbb{H}$, prove that its stabilizer $G_z = \{g \in \mathrm{SL}_2(\mathbb{R}) : gz = z\}$ is isomorphic to $\mathrm{SO}_2(\mathbb{R}) = \{g \in M_2(\mathbb{R}) : gg^t = 1\}$, where g^t is the transpose of g.

(c)(10 points) Let $\mathbb{Z}$ be the set of integers and let

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z}, \quad a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \pmod 2 \right\}$$

be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ (no need to prove this), and let it act on $\mathbb{Q} \cup \{\infty\}$ linearly fractionally as above. How many orbits does this action have? Give a representative for each orbit.

Problem 2. Let $p \geq 7$ be an odd prime number.

(a) (5 points) (to warm up) Evaluate the rational number $\cos(\pi/7) \cdot \cos(2\pi/7) \cdot \cos(3\pi/7)$.

(b) (15 points) Show that $\prod_{n=1}^{(p-1)/2} \cos(n\pi/p)$ is a rational number and determine its value.

Problem 3. (20 points, 10 points each) For any 3×3 matrix $A \in M_3(\mathbb{Q})$, let A^{db} be the 6×6 matrix

$$A^{db} := \begin{pmatrix} 0 & I_3 \\ A & 0 \end{pmatrix}$$

(a) Express the characteristic and minimal polynomials of A^{db} over $\mathbb{Q}$ in terms of the characteristic and minimal polynomial of A.

(b) Suppose that $A, B \in M_3(\mathbb{Q})$ are such that A^{db} and B^{db} are conjugate in the sense that there exists an element $C \in GL_6(\mathbb{Q})$ such that $C \cdot A^{db} \cdot C^{-1} = B^{db}$. Are A and B conjugate? (Either prove this statement or give a counterexample.)

Problem 4. (20 points) Classify all groups of order 8.

Problem 5. Let V be a finite dimensional vector space over complex field $\mathbb{C}$ with a non-degenerate symmetric bilinear form $(,)$. Let

$$O(V) = \{g \in \mathrm{GL}(V) | (gu, gv) = (u, v),\ u, v \in V\}$$

be the orthogonal group.

(a) (10 points) Prove that

$$(V \otimes_{\mathbb{C}} V)^{O(V)} \cong \mathrm{End}_{O(V)}(V),$$

and construct one such isomorphism. Here $O(V)$ acts on $V \otimes_{\mathbb{C}} V$ via $g(a \otimes b) = ga \otimes gb$, and $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is the fixed point subspace of $V \otimes V$.

(b) (10 points) Prove that the fixed point subspace $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is 1-dimensional.

Problem 6. (20 points) Let c be a non-zero rational integer.

(a) (6 points) Factorize the three variable polynomial

$$f(x, y, z) = x^3 + cy^3 + c^2z^3 - 3cxyz$$

over $\mathbb{C}$ (you may assume $c = \theta^3$ for some $\theta \in \mathbb{C}$).

(b) (7 points) When $c = \theta^3$ is a cube for some rational integer θ, prove that there are only finitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.

(c) (7 points) When c is not a cube of any rational integers, prove that there infinitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.

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Applied Math. and Computational Math.

Team

Please solve as many problems as you can!

1. (15 pts)
Given a finite positive (Borel) measure $d\mu$ on $[0,1]$, define its sequence of moments as follows

$$c_j = \int_0^1 x^j \, d\mu(x)\,, \quad j = 0, 1, \ldots .$$

Show that the sequence is *completely monotone* in the sense that that

$$(I - S)^k c_j \geq 0 \quad \text{for all } j, k \geq 0,$$

where S denotes the backshift operator given by $Sc_j = c_{j+1}$ for $j \geq 0$.

2. (20 pts)
We recall that a polynomial

$$f(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

is called an Eisenstein polynomial if for some prime p we have

(i) $p \mid a_i$ for $i = 0, \ldots, d-1$,
(ii) $p^2 \nmid a_0$,
(iii) $p \nmid a_d$.

Eisenstein polynomials are well-know to be irreducible over $\mathbb{Z}$, so they can be used to construct explicit examples of irreducible polynomials.

Questions:

(i) Prove that a composition $f(g(X))$ of two Eisenstein polynomials f and g is an Eisenstein polynomial again.
(ii) Suggest a multivariate generalisation of the Eisenstein polynomials. That is, describe a class polynomials $F(X_1, \ldots, X_m)$ in terms of the divisibility properties of their coefficients that are guaranteed to be irreducible.

3. (20 pts) For solving the following partial differential equation

$$u_t + f(u)_x = 0, \qquad 0 \leq x \leq 1 \tag{1}$$

where $f'(u) \geq 0$, with periodic boundary condition, we can use the following semi-discrete discontinuous Galerkin method: Find $u_h(\cdot, t) \in V_h$ such that, for all $v \in V_h$ and $j = 1, 2, \cdots, N$,

$$\int_{I_j} (u_h)_t v dx - \int_{I_j} f(u_h) v_x dx + f((u_h)^-_{j+1/2}) v^-_{j+1/2} - f((u_h)^-_{j-1/2}) v^+_{j-1/2} = 0, \tag{2}$$

with periodic boundary condition

$$(u_h)^-_{1/2} = (u_h)^-_{N+1/2}; \quad (u_h)^+_{N+1/2} = (u_h)^+_{1/2}, \tag{3}$$

where $I_j = (x_{j-1/2}, x_{j+1/2})$, $0 = x_{1/2} < x_{3/2} < \cdots < x_{N+1/2} = 1$, $h = \max_j(x_{j+1/2} - x_{j-1/2})$, $v^\pm_{j+1/2} = v(x^\pm_{j+1/2}, t)$, and

$$V_h = \{v : v|_{I_j} \text{ is a polynomial of degree at most } k \text{ for } 1 \le j \le N\}.$$

Prove the following L^2 stability of the scheme

$$\frac{d}{dt}E(t) \le 0 \tag{4}$$

where $E(t) = \int_0^1 (u_h(x,t))^2 dx$.

4. Consider the linear system $Ax = b$. The GMRES method is a projection method which obtains a solution in the m-th Krylov subspace K_m so that the residual is orthogonal to AK_m. Let r_0 be the initial residual and let $v_0 = r_0$. The Arnoldi process is applied to build an orthonormal system $v_1, v_2, \cdots, v_{m-1}$ with $v_1 = Av_0/\|Av_0\|$. The approximate solution is obtained from the following space

$$K_m = \text{span}\{v_0, v_1, \cdots, v_{m-1}\}.$$

(i) (5 points) Show that the approximate solution is obtained as the solution of a least-square problem, and that this problem is triangular.
(ii) (5 points) Prove that the residual r_k is orthogonal to $\{v_1, v_2, \cdots, v_{k-1}\}$.
(iii) (5 points) Find a formula for the residual norm.
(iv) (5 points) Derive the complete algorithm.

5. (10 pts)

(i) Set $x_0 = 0$. Write the recurrence

$$x_k = 2x_{k-1} + b_k, \quad k = 1, 2, \cdots, n,$$

in a matrix form $A\vec{x} = \vec{b}$. For $b_1 = -1/3$, $b_k = (-1)^k, k = 2, 3, \cdots, n$, verify that $x_k = (-1)^k/3, k = 1, 2, \cdots, n$ is the exact solution.
(ii) Find A^{-1} and compute condition number of A in L^1 norm.