Please solve 5 out of the following 6 problems.

1. Calculate the integral:
   \[ \int_0^\infty \frac{\log x}{1 + x^2} \, dx. \]

2. Construct an increasing function on \( \mathbb{R} \) whose set of discontinuities is precisely \( \mathbb{Q} \).

3. Prove that any bounded analytic function \( F \) over \( \{ z \mid r < |z| < R \} \) can be written as \( F(z) = z^\alpha f(z) \), where \( f \) is an analytic function over the disk \( \{ z \mid |z| < R \} \) and \( \alpha \) is a constant.

4. Let \( D \subset \mathbb{R}^n \) be a bounded open set, \( f : \bar{D} \to \bar{D} \) is a smooth map such that its Jacobian \( \left| \frac{\partial f}{\partial x} \right| \equiv 1 \), where \( \bar{D} \) denotes the closure of \( D \).
   
   Prove
   
   (a) for each small ball \( B_\varepsilon(x) \), there exists a positive integer \( k \) such that \( f^k(B_\varepsilon(x)) \cap B_\varepsilon(x) \neq \emptyset \), where \( B_\varepsilon(x) \) denotes the ball centered at \( x \) with radius \( \varepsilon \);
   
   (b) there exists \( x \in \bar{D} \) and a sequence \( k_1, k_2, \ldots, k_j, \ldots \) such that \( f^{k_j}(x) \to x \) as \( k_j \to \infty \).

5. Let \( u \) be a subharmonic function over a domain \( \Omega \subset \mathbb{C} \), i.e., it is twice differentiable and \( \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0 \). Prove that \( u \) achieves its maximum in the interior of \( \Omega \) only when \( u \) is a constant.

6. Suppose that \( \phi \in C_0^\infty(\mathbb{R}^n) \), \( \int_{\mathbb{R}^n} \phi \, dx = 1 \). Let \( \phi_\varepsilon(x) = \varepsilon^{-n} \phi(x/\varepsilon), x \in \mathbb{R}^n, \varepsilon > 0 \). Prove that if \( f \in L^p(\mathbb{R}^n), 1 \leq p < \infty \), then \( f \ast \phi_\varepsilon \to f \) in \( L^p(\mathbb{R}^n) \), as \( \varepsilon \to 0 \). It is not true for \( p = \infty \).
Problem 1. Suppose that $X_n$ converges to $X$ in distribution and $Y_n$ converges to a constant $c$ in distribution. Show that

(a) $Y_n$ converges to $c$ in probability;
(b) $X_nY_n$ converges to $cX$ in distribution.

Problem 2. Let $X$ and $Y$ be two random variables with $|Y| > 0$, a.s. Let $Z = X/Y$.

(a) Assume the distribution function of $(X, Y)$ has the density $p(x, y)$. What is the density function of $Z$?
(b) Assume $X$ and $Y$ are independent and $X$ is $N(0, 1)$ distributed, $Y$ has the uniform distribution on $(0, 1)$. Give the density function of $Z$.

Problem 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) Let $\mathcal{G}$ be a sub $\sigma$-algebra of $\mathcal{F}$, and $\Gamma \in \mathcal{F}$. Prove that the following properties are equivalent:

(i) $\Gamma$ is independent of $\mathcal{G}$ under $P$,
(ii) for every probability $Q$ on $(\Omega, \mathcal{F})$, equivalent to $P$, with $dQ/dP$ being $\mathcal{G}$ measurable, we have $Q(\Gamma) = P(\Gamma)$.

(b) Let $X, Y, Z$ be random variables and $Y$ is integrable. Show that if $(X, Y)$ and $Z$ are independent, then $E[Y|X, Z] = E[Y|X]$.

Problem 4. Let $X_1, ..., X_n$ be i.i.d. $N(0, \sigma^2)$, and let $M$ be the mean of $|X_1|, ..., |X_n|$.

1. Find $c \in R$ so that $\hat{\sigma} = cM$ is a consistent estimator of $\sigma$.
2. Determine the limiting distribution for $\sqrt{n}(\hat{\sigma} - \sigma)$.
3. Identify an approximate $(1 - \alpha)\%$ confidence interval for $\sigma$.
4. Is $\hat{\sigma} = cM$ asymptotically efficient? Please justify your answer.
**Problem 5.** The shifted exponential distribution has the density function

\[ f(y; \phi, \theta) = \frac{1}{\theta} \exp\left\{-\left(\frac{u - \phi}{\theta}\right)\right\}, \quad y > \phi, \theta > 0. \]

Let \( Y_1, \ldots, Y_n \) be a random sample from this distribution. Find the maximum likelihood estimator (MLE) of \( \phi \) and \( \theta \) and the limiting distribution of the MLE.

You may use the following Rényi representation of the order statistics: Let \( E_1, \ldots, E_n \), be a random sample from the standard exponential distribution (i.e., the above distribution with \( \phi = 0, \theta = 1 \)). Let \( E_{(r)} \) denote the \( r \)-th order statistics. According to the Rényi representation,

\[ E_{(r)} \overset{D}{=} \sum_{j=1}^{r} \frac{E_j}{n+1-j}, \quad r = 1, \ldots, n. \]

Here, the symbol \( \overset{D}{=} \) denotes equal in distribution.
1. Compute the fundamental and homology groups of the wedge sum of a circle $S^1$ and a torus $T = S^1 \times S^1$.

2. Given a properly discontinuous action $F : G \times M \to M$ on a smooth manifold $M$, show that $M/G$ is orientable if and only if $M$ is orientable and $F(g, \cdot)$ preserves the orientation of $M$. Use this statement to show that the Möbius band is not orientable and that $\mathbb{R}P^n$ is orientable if and only if $n$ is odd.

3. (a) Consider the space $Y$ obtained from $S^2 \times [0, 1]$ by identifying $(x, 0)$ with $(-x, 0)$ and also identifying $(x, 1)$ with $(-x, 1)$, for all $x \in S^2$. Show that $Y$ is homeomorphic to the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

   (b) Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

4. Prove that a bi-invariant metric on a Lie group $G$ has nonnegative sectional curvature.

5. Let $M$ be the upper half-plane $\mathbb{R}^2_+$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^k}.$$ 

For which values of $k$ is $M$ complete?

6. Given any nonorientable manifold $M$ show the existence of a smooth orientable manifold $\overline{M}$ which is a double covering of $M$. Find $\overline{M}$ when $M$ is $\mathbb{R}P^2$ or the Möbius band.
Problem 1. Let the special linear group (of order 2)

\[ \text{SL}_2(\mathbb{R}) = \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : \det g = 1 \} \]

act on the upper half plane \( \mathbb{H} = \{ z = x + iy \in \mathbb{C} : y > 0 \} \) linear fractionally:

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d} \]

(a) (5 points) Prove that the action is transitive, i.e., for any two \( z_1, z_2 \in \mathbb{H} \), there is \( g \in \text{SL}_2(\mathbb{R}) \) such that \( gz_1 = z_2 \).

(b) (5 points) For a fixed \( z \in \mathbb{H} \), prove that its stabilizer \( G_z = \{ g \in \text{SL}_2(\mathbb{R}) : gz = z \} \) is isomorphic to \( \text{SO}_2(\mathbb{R}) = \{ g \in M_2(\mathbb{R}) : gg^t = 1 \} \), where \( g^t \) is the transpose of \( g \).

(c) (10 points) Let \( \mathbb{Z} \) be the set of integers and let \( \Gamma(2) = \{ (a b) \in \text{SL}_2(\mathbb{R}) : a,b,c,d \in \mathbb{Z}, a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \pmod{2} \} \) be a discrete subgroup of \( \text{SL}_2(\mathbb{R}) \) (no need to prove this), and let it act on \( \mathbb{Q} \cup \{ \infty \} \) linearly fractionally as above. How many orbits does this action have? Give a representative for each orbit.

Problem 2. Let \( p \geq 7 \) be an odd prime number.

(a) (5 points) (to warm up) Evaluate the rational number \( \cos(\pi/7) \cdot \cos(2\pi/7) \cdot \cos(3\pi/7) \).

(b) (15 points) Show that \( \prod_{n=1}^{(p-1)/2} \cos(n\pi/p) \) is a rational number and determine its value.

Problem 3. (20 points, 10 points each) For any \( 3 \times 3 \) matrix \( A \in M_3(\mathbb{Q}) \), let \( A^{db} \) be the \( 6 \times 6 \) matrix

\[ A^{db} := \begin{pmatrix} 0 & I_3 \\ A & 0 \end{pmatrix} \]

(a) Express the characteristic and minimal polynomials of \( A^{db} \) over \( \mathbb{Q} \) in terms of the characteristic and minimal polynomial of \( A \).

(b) Suppose that \( A, B \in M_3(\mathbb{Q}) \) are such that \( A^{db} \) and \( B^{db} \) are conjugate in the sense that there exists an element \( C \in GL_6(\mathbb{Q}) \) such that \( C \cdot A^{db} \cdot C^{-1} = B^{db} \). Are \( A \) and \( B \) conjugate? (Either prove this statement or give a counterexample.)

Problem 4. (20 points) Classify all groups of order 8.
Problem 5. Let $V$ be a finite dimensional vector space over complex field $\mathbb{C}$ with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$. Let

$$O(V) = \{ g \in \text{GL}(V) | (gu, gv) = (u, v), \; u, v \in V \}$$

be the orthogonal group.

(a) (10 points) Prove that

$$(V \otimes \mathbb{C} V)^{O(V)} \cong \text{End}_{O(V)}(V),$$

and construct one such isomorphism. Here $O(V)$ acts on $V \otimes \mathbb{C} V$ via $g(a \otimes b) = ga \otimes gb$, and $(V \otimes \mathbb{C} V)^{O(V)}$ is the fixed point subspace of $V \otimes V$.

(b) (10 points) Prove that the fixed point subspace $(V \otimes \mathbb{C} V)^{O(V)}$ is 1-dimensional.

Problem 6. (20 points) Let $c$ be a non-zero rational integer.

(a) (6 points) Factorize the three variable polynomial

$$f(x, y, z) = x^3 + cy^3 + c^2z^3 - 3xyz$$

over $\mathbb{C}$ (you may assume $c = \theta^3$ for some $\theta \in \mathbb{C}$).

(b) (7 points) When $c = \theta^3$ is a cube for some rational integer $\theta$, prove that there are only finitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.

(c) (7 points) When $c$ is not a cube of any rational integers, prove that there infinitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$. 
1. (15 pts)
Given a finite positive (Borel) measure $d\mu$ on $[0, 1]$, define its sequence of moments as follows
$$c_j = \int_0^1 x^j d\mu(x), \quad j = 0, 1, \ldots.$$ 
Show that the sequence is completely monotone in the sense that that
$$(I - S)^k c_j \geq 0 \quad \text{for all } j, k \geq 0,$$
where $S$ denotes the backshift operator given by $Sc_j = c_{j+1}$ for $j \geq 0$.

2. (20 pts)
We recall that a polynomial
$$f(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$
is called an Eisenstein polynomial if for some prime $p$ we have
(i) $p \mid a_i$ for $i = 0, \ldots, d - 1$, 
(ii) $p^2 \nmid a_0$, 
(iii) $p \nmid a_d$.

Eisenstein polynomials are well-known to be irreducible over $\mathbb{Z}$, so they can be used to construct explicit examples of irreducible polynomials.

Questions:
(i) Prove that a composition $f(g(X))$ of two Eisenstein polynomials $f$ and $g$ is an Eisenstein polynomial again.
(ii) Suggest a multivariate generalisation of the Eisenstein polynomials. That is, describe a class polynomials $F(X_1, \ldots, X_m)$ in terms of the divisibility properties of their coefficients that are guaranteed to be irreducible.

3. (20 pts) For solving the following partial differential equation
$$u_t + f(u)_x = 0, \quad 0 \leq x \leq 1 \quad (1)$$
where $f'(u) \geq 0$, with periodic boundary condition, we can use the following semi-discrete discontinuous Galerkin method: Find $u_h(\cdot, t) \in V_h$ such that, for all $v \in V_h$ and $j = 1, 2, \cdots, N$,
$$\int_{I_j} (u_h)_t v dx - \int_{I_j} f(u_h) v_x dx + f((u_h)^{+}_{j+1/2}) v^-_{j+1/2} - f((u_h)^{-}_{j-1/2}) v^+_{j-1/2} = 0, \quad (2)$$
with periodic boundary condition
\[ (u_h)_{1/2}^N = (u_h)_{N+1/2}; \quad (u_h)^+_{N+1/2} = (u_h)^+_{1/2}, \quad (3) \]
where \( I_j = (x_j-1/2, x_{j+1/2}) \), \( 0 = x_{1/2} < x_{3/2} < \cdots < x_{N+1/2} = 1 \),
\( h = \max_j (x_{j+1/2} - x_{j-1/2}) \), \( v_j^\pm = v(x_j^\pm, t) \), and
\[ V_h = \{ v : v|_{I_j} \text{ is a polynomial of degree at most } k \text{ for } 1 \leq j \leq N \}. \]

Prove the following \( L^2 \) stability of the scheme
\[ \frac{d}{dt} E(t) \leq 0 \quad (4) \]
where \( E(t) = \int_0^1 (u_h(x, t))^2 dx \).

4. Consider the linear system \( Ax = b \). The GMRES method is a projection method which obtains a solution in the \( m \)-th Krylov subspace \( K_m \) so that the residual is orthogonal to \( AK_m \). Let \( r_0 \) be the initial residual and let \( v_0 = r_0 \). The Arnoldi process is applied to build an orthonormal system \( v_1, v_2, \cdots, v_{m-1} \) with \( v_1 = Av_0/\|Av_0\| \). The approximate solution is obtained from the following space
\[ K_m = \text{span}\{v_0, v_1, \cdots, v_{m-1}\}. \]

(i) (5 points) Show that the approximate solution is obtained as the solution of a least-square problem, and that this problem is triangular.

(ii) (5 points) Prove that the residual \( r_k \) is orthogonal to \( \{v_1, v_2, \cdots, v_{k-1}\} \).

(iii) (5 points) Find a formula for the residual norm.

(iv) (5 points) Derive the complete algorithm.

5. (10 pts)
(i) Set \( x_0 = 0 \). Write the recurrence
\[ x_k = 2x_{k-1} + b_k, \quad k = 1, 2, \cdots, n, \]
in a matrix form \( A\vec{x} = \vec{b} \). For \( b_1 = -1/3 \), \( b_k = (-1)^k, k = 2, 3, \cdots, n \), verify that \( x_k = (-1)^k/3, k = 1, 2, \cdots, n \) is the exact solution.

(ii) Find \( A^{-1} \) and compute condition number of \( A \) in \( L^1 \) norm.