The Work of Martin Hairer

Martin Hairer has made a major breakthrough in the study of stochastic partial differential equations by creating a new theory that provides tools for attacking problems that up to now had seemed impenetrable.

The subject of differential equations has its roots in the development of calculus by Isaac Newton and Gottfried Leibniz in the 17th century. A major motivation at that time was to understand the motion of the planets in the solar system. Newton's laws of motion can be used to formulate a differential equation that describes, for example, the motion of the Earth around the Sun. A *solution* to such an equation is a function that gives the position of the Earth at any time t. In the centuries since, differential equations have become ubiquitous across all areas of science and engineering to describe systems that change over time.

A differential equation describing planetary motion is *deterministic*, meaning that it determines exactly where a planet will be at a given time in the future. Other differential equations are *stochastic*, meaning that they describe systems containing an inherent element of randomness. An example is an equation that describes how a stock price will change over time. Such an equation incorporates a term that represents fluctuations in the stock market price. If one could predict exactly what the fluctuations would be, one could predict the future stock price exactly (and get very rich!). However, the fluctuations, while having some dependence on the initial stock price, are essentially random and unpredictable. The stock-price equation is an example of a *stochastic* differential equation.

In the planetary-motion equation, the system changes with respect to only one variable, namely, time. Such an equation is called an *ordinary* differential equation (ODE). By contrast, *partial* differential equations (PDEs) describe systems that change with respect to more than one variable, for example, time and position. Many PDEs are *nonlinear*, meaning that the terms in it are not simple proportions—for example, they might be raised to an exponential power. Some of the most important natural phenomena are governed by nonlinear PDEs, so understanding these equations is a major goal for mathematics and the sciences. However, nonlinear PDEs are among the most difficult mathematical objects to understand. Hairer's work has caused a great deal of excitement because it develops a general theory that can be applied to a large class of nonlinear stochastic PDEs.

An example of a nonlinear stochastic PDE—and one that played an important role in Hairer's work—is the KPZ equation, which is named for Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang, the physicists who came up with the equation in 1986. The KPZ equation describes the evolution over time of the interface between two substances. To get a feel for the nature of this equation, consider the process of liquid crystal display manufacturing. In a simplified model of this process, one imagines drops of the liquid-crystal material being deposited between two closely aligned vertical sheets of glass. The drops interact with other drops, adhering to each other, merging, and spreading out as they settle to the bottom. At a scale much smaller than the scale at which one views this process, the molecules in the drops move in a random way. One can think of this random motion as introducing "white noise" into the system. It creates a rough, irregular interface between the air above and the material accumulating below. The KPZ equation describes the evolution of this interface over time. Because it includes a white-noise term to describe the random motion of the molecules, the KPZ equation is a stochastic PDE. A solution to the KPZ equation would provide, for any time t and any point along the bottom edge of the glass, the height of the interface above that point.

The challenge the KPZ equation posed is that, although it made sense from the point of view of physics, it did not make sense mathematically. A solution to the KPZ equation should be a mathematical object that represents the rough, irregular nature of the interface. Such an object has no smoothness; in mathematical terms, it is not *differentiable.* And yet two of the terms in the KPZ equation call for the object to be differentiable. There is a way to sidestep this difficulty by using an object called a *distribution.* But then a new problem arises, because the KPZ equation is nonlinear: It contains a square term, and distributions cannot be squared. For these reasons, the KPZ equation was not well defined. Although researchers came up with some technical tricks to ameliorate these difficulties for the special case of the KPZ equation, the fundamental problem of its not being well defined long remained an unresolved issue.

In a spectacular achievement, Hairer overcame these difficulties by describing a new approach to the KPZ equation that allows one to give a mathematically precise meaning to the equation and its solutions. What is more, in subsequent work he used the ideas he developed for the KPZ equation to build a general theory, *the theory of regularity structures*, that can be applied to a broad class of stochastic PDEs. In particular, Hairer's theory can be used in higher dimensions (the KPZ equation has one spatial dimension because it models an idealization of the interface as a one-dimensional curve).

The basic idea of Hairer's approach to the KPZ equation is the following. Instead of making the usual assumption that the small random effects occur on an infinitesimally small scale, he adopted the assumption that the random effects occur on a scale that is small in comparison to the scale at which the system is viewed. Removing the infinitesimal assumption, which Hairer calls "regularizing the noise", renders an equation that can be solved. The resulting solution is *not* a solution to KPZ; rather, it can be used as the starting point to construct a sequence of objects that, in the limit, converges to a solution of KPZ. And Hairer proved a crucial fact: the limiting solution is always the same regardless of the kind of noise regularization that is used.

Hairer's general theory addresses other, higher-dimensional stochastic PDEs that are not well defined. For these equations, as with KPZ, the main challenge is that, at very small scales, the behavior of the solutions is very rough and irregular. If the solution were a smooth function, one could carry out a *Taylor expansion*, which is a way of approximating the function by polynomials of increasingly higher degree. But the roughness of the solutions means they are not well approximated by polynomials. What Hairer did instead is to define objects, custom-built for the equation at hand, that approximate the behavior of the solution at small scales. These objects then play a role similar to polynomials in a Taylor expansion. At each point, the solution will look like an infinite superposition of these objects. The ultimate solution is then obtained by gluing together the pointwise superpositions. Hairer established the crucial fact that the ultimate solution does not depend on the approximating objects used to obtain it.

Prior to Hairer's work, researchers had made a good deal of progress in understanding *linear* stochastic PDEs, but there was a fundamental block to addressing nonlinear cases. Hairer's new theory goes a long way towards removing that block. What is more, the class of equations to which the theory applies contains several that are of central interest in mathematics and science. In addition, his work could open the way to understanding the phenomenon of universality. Other equations, when rescaled, converge to the KPZ equation, so there seems to be some universal phenomenon lurking in the background. Hairer's work has the potential to provide rigorous analytical tools to study this universality.

Before developing the theory of regularity structures, Hairer made other outstanding contributions. For example, his joint work with Jonathan Mattingly constitutes a significant advance in understanding a stochastic version of the Navier-Stokes equation, a nonlinear PDE that describes wave motion.

In addition to being one of the world's top mathematicians, Hairer is a very good computer programmer. While still a school student, he created audio editing software that he later developed and successfully marketed as "the Swiss army knife of sound editing". His mathematical work does not depend on computers, but he does find that programming small simulations helps develop intuition.

With his commanding technical mastery and deep intuition about physical systems, Hairer is a leader in the field who will doubtless make many further significant contributions.

References

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Biography

Born in 1975, Martin Hairer is an Austrian citizen. In 2001, he received his PhD in physics from the University of Geneva, under the direction of Jean-Pierre Eckmann. He is currently Regius Professor of Mathematics at the University of Warwick. His honors include the Whitehead Prize of the London Mathematical Society (2008), the Philip Leverhulme Prize (2008), the Wolfson Research Merit Award of the Royal Society (2009), the Fermat Prize (2013), and the Fröhlich Prize of the London Mathematical Society (2014). He was elected a Fellow of the Royal Society in 2014.