The 7^{th} Romanian Master of Mathematics Competition

Day 1: Friday, February 27, 2015, Bucharest

Language: English

Problem 1. Does there exist an infinite sequence of positive integers a_1, a_2, a_3, $\ldots$ such that a_m and a_n are coprime if and only if $|m - n| = 1$?

Problem 2. For an integer $n \geq 5$, two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is *legal* if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

Problem 3. A finite list of rational numbers is written on a blackboard. In an *operation*, we choose any two numbers a, b, erase them, and write down one of the numbers

$$a + b,\ a - b,\ b - a,\ a \times b,\ a/b \text{ (if } b \neq 0),\ b/a \text{ (if } a \neq 0).$$

Prove that, for every integer $n > 100$, there are only finitely many integers $k \geq 0$, such that, starting from the list

$$k + 1,\ k + 2,\ \ldots,\ k + n,$$

it is possible to obtain, after $n - 1$ operations, the value $n!$.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.