1. Let $\phi \in C([a, b], R)$. Suppose for every function $h \in C^1([a, b], R)$, $h(a) = h(b) = 0$, we have

$$\int_a^b \phi(x)h(x)dx = 0.$$ 

Prove that $\phi(x) = 0$.

2. Let $f$ be a Lebesgue integrable function over $[a, b + \delta], \delta > 0$, prove that

$$\lim_{h \to 0^+} \int_a^b |f(x + h) - f(x)|dx \to 0.$$ 

3. Let $L(q, q', t)$ be a function of $(q, q', t) \in T U \subset R^n$. Let $\gamma : [a, b] \to U$ be a curve in $U$. Define a functional $S(\gamma) = \int_a^b L(\gamma(t), \gamma'(t), t)dt$. We say that $\gamma$ is an extremal if for every smooth variation of $\gamma$, $\phi(t, s), s \in (-\delta, \delta), \phi(t, 0) = \gamma(t), \phi_s = \phi(t, s)$, we have $\frac{dS(\phi_s)}{ds}|_{s=0} = 0$. Prove that every extremal $\gamma$ satisfies the Euler-Lagrange equation: $\frac{d}{dt}(\frac{\partial L}{\partial q'}) = \frac{\partial L}{\partial q}$.

4. Let $f : U \to U$ be a holomorphic function with $U$ a bounded domain in the complex plane. Assuming $0 \in U, f(0) = 0, f'(0) = 1$, prove that $f(z) = z$.

5. Let $T : H_1 \to H_2$ be a bounded operator of Hilbert spaces $H_1, H_2$. Let $S : H_1 \to H_2$ be a compact operator, that is, for every bounded sequence $\{v_n\} \in H_1, Sv_n$ has a converging subsequence. Show that $\text{Coker}(T + S) = H_2/\text{Im}(T + S)$ is finite dimensional and $\text{Im}(T + S)$ is closed in $H_2$. (Hint: Consider equivalent statements in terms of adjoint operators.)

6. Let $u \in C^2(\Omega), \Omega \subset R^d$ is a bounded domain with a smooth boundary.

1) Let $u$ be a solution of the equation $\Delta u = f, u|_{\partial \Omega} = 0, f \in L^2(\Omega)$. Prove that there is a constant $C$ depends only $\Omega$ such that

$$\int_{\Omega} (\sum_{j=1}^n (\frac{\partial u}{\partial x_j})^2 + u^2)dx \leq C \int_{\Omega} f^2(x)dx.$$ 

2) Let $\{u_n\}$ be a sequence of harmonic functions on $\Omega$, such that $||u_n||_{L^2(\Omega)} \leq M < \infty$, for a constant $M$ independent of $n$. Prove that there is a converging subsequence $\{u_{n_k}\}$ in $L^2(\Omega)$. 

**Problem 1.** One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

**Problem 2.** Assume a sequence of random variables $X_n$ converges in distribution to a random variable $X$. Let $\{N_t, t \geq 0\}$ be a set of positive integer-valued random variables, which is independent of $(X_n)$ and converges in probability to $\infty$ as $t \to \infty$. Prove that $X_{N_t}$ converges in distribution to $X$ as $t \to \infty$.

**Problem 3.** Suppose $T_1, T_2, \ldots, T_n$ is a sequence of independent, identically distributed random variables with the exponential distribution of the density function

$$p(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

Let $S_n = T_1 + T_2 + \cdots + T_n$. Find the distribution of the random vector

$$V_n = \left\{ \frac{T_1}{S_n}, \frac{T_2}{S_n}, \ldots, \frac{T_n}{S_n} \right\}.$$  

**Problem 4.** Suppose that $X$ and $Z$ are jointly normal with mean zero and standard deviation 1. For a strictly monotonic function $f(\cdot)$, $\text{cov}(X, Z) = 0$ if and only if $\text{cov}(X, f(Z)) = 0$, provided the latter covariance exists. **Hint:** $Z$ can be expressed as $Z = \rho X + \varepsilon$ where $X$ and $\varepsilon$ are independent and $\varepsilon \sim N(0, \sqrt{1-\rho^2})$.

**Problem 5.** Consider the following penalized least-squares problem (Lasso):

$$\frac{1}{2} \|Y - X\beta\|^2 + \lambda \|\beta\|_1$$

Let $\hat{\beta}$ be a minimizer and $\Delta = \hat{\beta} - \beta^*$ for any given $\beta^*$. If $\lambda > 2\|X^T(Y - X\beta^*)\|_\infty$, show that

1. $\|Y - X^T\hat{\beta}\|^2 - \|Y - X^T\beta^*\|^2 > -\lambda \|\Delta\|_1$.  

S.-T. Yau College Student Mathematics Contests 2015

**Probability and Statistics**

**Team (5 problems)**
2. $\|\Delta_S^c\|_1 \leq 3\|\Delta_S\|_1$, where $S = \{ j : \beta_j^* \neq 0 \}$ is the support of the vector $\beta^*$, $S^c$ is its complement set, $\Delta_S$ is the subvector of $\Delta$ restricted on the set $S$, and $\|\Delta_S\|_1$ is its $L_1$-norm.
1. Let $SO(3)$ be the set of all $3 \times 3$ real matrices $A$ with determinant 1 and satisfying $^tAA = I$, where $I$ is the identity matrix and $^tA$ is the transpose of $A$. Show that $SO(3)$ is a smooth manifold, and find its fundamental group. You need to prove your claims.

2. Let $X$ be a topological space. The suspension $S(X)$ of $X$ is the space obtained from $X \times [0, 1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of $X$ and $S(X)$.

3. Let $F : M \to N$ be a smooth map between two manifolds. Let $X_1, X_2$ be smooth vector fields on $M$ and let $Y_1, Y_2$ be smooth vector fields on $N$. Prove that if $Y_1 = F_*X_1$ and $Y_2 = F_*X_2$, then $F_*[X_1, X_2] = [Y_1, Y_2]$, where $[,]$ is the Lie bracket.

4. Let $M_1$ and $M_2$ be two compact convex closed surfaces in $\mathbb{R}^3$, and $f : M_1 \to M_2$ a diffeomorphism such that $M_1$ and $M_2$ have the same inner normal vectors and Gauss curvatures at the corresponding points. Prove that $f$ is a translation.

5. Prove the second Bianchi identity:
   \[ R_{ijkl,h} + R_{ijlh;k} + R_{ijhk;l} = 0 \]

6. Let $M_1, M_2$ be two complete $n$-dimensional Riemannian manifolds and $\gamma_i : [0, a] \to M_i$ are two arc length parametrized geodesics. Let $\rho_i$ be the distance function to $\gamma_i(0)$ on $M_i$. Assume that $\gamma_i(a)$ is within the cut locus of $\gamma_i(0)$ and for any $0 \leq t \leq a$ we have the inequality of sectional curvatures
   \[ K_1(X_1, \frac{\partial}{\partial \gamma_1}) \geq K_2(X_2, \frac{\partial}{\partial \gamma_2}), \]
   where $X_i \in T_{\gamma_i(t)}M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}$.
   Then
   \[ Hess(\rho_1)(\widetilde{X}_1, \widetilde{X}_1) \leq Hess(\rho_2)(\widetilde{X}_2, \widetilde{X}_2), \]
   where $\widetilde{X}_i \in T_{\gamma_i(a)}M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}(a)$. 

Problem 1. (20pt) Let $V = \mathbb{R}^n$ be an Euclidean space equipped with usual inner product, and $g$ an orthogonal matrix acting on $V$. For $a \in V$, let $s_a$ denote the reflection
\[ s_a(x) := x - 2\frac{(x, a)}{(a, a)}a, \quad \forall x \in V. \]

(1.1) (10pt) For $a = (g - 1)b \neq 0$, show that
\[ \ker(s_aga - 1) = \ker(g - 1) \oplus \mathbb{R}b. \]

(1.2) (10pt) Show that $g$ is a product of $\dim((g - 1)V)$ reflections.

Problem 2. (20pt) Let $p$ and $q$ be two distinct prime numbers. Let $G$ be a non-abelian finite group satisfying the following conditions:

1. all nontrivial elements have order either $p$ or $q$;
2. The $q$-Sylow subgroup $H_q$ is normal and is a nontrivial abelian group.

Show in steps the following statement:

The group $G$ is of the form $(\mathbb{Z}/p\mathbb{Z}) \rtimes (\mathbb{Z}/q\mathbb{Z})^n$, where the action of $1 \in \mathbb{Z}/p\mathbb{Z}$ on $(\mathbb{Z}/q\mathbb{Z})^n \simeq \mathbb{F}_q^n$ is given by a matrix $M(1) \in \text{GL}_n(\mathbb{F}_q)$ whose eigenvalues are all primitive $p$-th roots of unities.

(2.1) (5pt) Let $H_p$ denote a $p$-Sylow subgroup of $G$. Show that its inclusion into $G$ induces an isomorphism $H_p \cong G/H_q$, and that $G \cong H_p \rtimes H_q$.

(2.2) (5pt) Let $M : H_p \to \text{Aut}(H_q) \simeq \text{GL}_n(\mathbb{F}_q)$ be the homomorphism induced from the conjugations. Show that for each $1 \neq a \in H_p$, $M(a)$ is semisimple whose eigenvalues are all primitive $p$-th roots of unities. In particular $M$ is injective.

(2.3) (5pt) Show that if two nontrivial elements $a, b \in H_p$ commute with each other, then $a = b^n$ for some $n \in \mathbb{N}$, and that $H_p \simeq \mathbb{Z}/p\mathbb{Z}$.

(2.4) (5pt) Complete the solution of the problem.
**Problem 3.** (20pt) Let $\zeta$ be a root of unity satisfying an equation $\zeta = 1 + N\eta$ for an integer $N \geq 3$ and an algebraic integer $\eta$. Show that $\zeta = 1$.

**Problem 4.** (20pt) Let $G$ be a finite group and $(\pi, V)$ a finite dimensional $CG$-module. For $n \geq 0$, let $\mathbb{C}[V]_n$ be the space of homogeneous polynomial functions on $V$ of degree $n$. For a simple $G$-representation $\rho$, denote by $a_n(\rho)$ the multiplicity of $\rho$ in $\mathbb{C}[V]_n$. Show that

$$
\sum_{n \geq 0} a_n(\rho)t^n = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\rho(g)} \det(\text{id}_V - \pi(g)t).
$$

**Problem 5.** (20pt) Let $A$ be an $n \times n$ complex matrix considered as an operator on $V = (\mathbb{C}^n, (\cdot, \cdot))$ with standard hermitian form. Let $A^* = \overline{A}^t$ be the hermitian transpose of $A$:

$$(Ax, y) = (x, A^*y), \quad \forall x, y \in \mathbb{C}^n.$$  

(5.1) (5pt) For any $\lambda \in \mathbb{C}$, show the identity:

$$\ker(A - \lambda)^\perp = (A^* - \overline{\lambda})V.$$  

(5.2) (15pt) Show the equivalence of the following two statements:

(a) $A$ commutes with $A^*$;  

(b) there is a unitary matrix $U$ (in the sense $U^* = U^{-1}$), such that $UAU^{-1}$ is diagonal.

**Problem 6.** (20pt) Consider the polynomial $f(x) = x^5 - 80x + 5$.

(6.1) (5pt) Show that $f$ is irreducible over $\mathbb{Q}$;

(6.2) (15 pt) Show in steps that the split field $K$ of $f$ has Galois group $G := \text{Gal}(K/\mathbb{Q})$ isomorphic to $S_5$, the symmetric group of 5 letters.

(a) (5pt) $f = 0$ has exactly two complex roots;  

(b) (5pt) $G$ can be embedded into $S_5$ with image containing cycles $(12345)$ and $(12)$;  

(c) (5pt) $G \simeq S_5$.  

Problem 1. Consider the elliptic interface problem

\[(a(x)u_x)_x = f, \ x \in (0, 1)\]

with the Dirichlet boundary condition

\[u(0) = u(1) = 0.\]

Here, \(f\) is a smooth function, the elliptic coefficient \(a(x)\) is discontinuous across an interface point \(\xi\), that is,

\[a(x) = \begin{cases} a_0 & \text{for } 0 < x < \xi \\ a_1 & \text{for } \xi < x < 1, \end{cases}\]

\(a_0, a_1 > 0\) are positive constants, and \(0 < \xi < 1\) is an interface point. Across the interface, we need to impose two jump conditions

\[u(\xi^-) = u(\xi^+), \ a(\xi^-)u_x(\xi^-) = a(\xi^+)u_x(\xi^+).\]

**Question:**

1. (25%) Design a numerical method to solve this problem. The method should be at least first order. It is better to be high order (if your method is first order, you get 20% points).

2. (75%) Prove your accuracy and convergence arguments (if your method is first order, you get 60% points).

Problem 2. Let \(G\) be graph of a social network, where for each pair of members there is either no connection, or a positive or a negative one.

An unbalanced cycle in \(G\) is a cycle which have odd number of negative edges.

Traversing along such a cycle with social rules such as friend of enemy are enemy would result in having a negative relation of one with himself!

A resigning in \(G\) at a vertex \(v\) of \(G\) is to switch the type (positive or negative) of all edges incident to \(v\).

**Question:** Show that one can switch all edge of \(G\) into positive edges using a sequence resigning if and only if there is no unbalanced cycle in \(G\).

Problem 3. We consider particles which are able to produce new particles of like kind. A single particle forms the original, or zero, generation. Every particle has probability \(p_k (k = 0, 1, 2, \ldots)\) of creating exactly \(k\) new particles; the direct descendants of the \(n\)th generation form the \((n + 1)\)st generation. The particles of each generation act independently of each other.
Assume $0 < p_0 < 1$. Let $P(x) = \sum_{k \geq 0} p_k x^k$ and $\mu = P'(1) = \sum_{k \geq 0} kp_k$ be the expected number of direct descendants of one particle. Prove that if $\mu > 1$, then the probability $x_n$ that the process terminates at or before the $n$th generation tends to the unique root $\sigma \in (0, 1)$ of equation $\sigma = P(\sigma)$.

**Problem 4.** (Isopermetric inequality). Consider a closed plane curve described by a parametric equation $(x(t), y(t)), 0 \leq t \leq T$ with parameter $t$ oriented counterclockwise and $(x(0), y(0)) = (x(T), y(T))$.

(a): Show that the total length of the curve is given by

$$L = \int_0^T \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

(b): Show that the total area enclosed by the curve is given by

$$A = \frac{1}{2} \int_0^T (x(t)y'(t) - y(t)x'(t)) \, dt$$

(c): The classical iso-perimetric inequality states that for closed plane curves with a fixed length $L$, circles have the largest enclosed area $A$. Formulate this question into a variational problem.

(d): Derive the Euler-Lagrange equation for the variational problem in (c).

(e): Show that there are two constants $x_0$ and $y_0$ such that

$$(x(t) - x_0)^2 + (y(t) - y_0)^2 \equiv r^2$$

where $r = L/(2\pi)$. Explain your result.

**Problem 5.** Let $A \in \mathbb{R}^{n \times m}$ with rank $r < \min(m, n)$. Let $A = U \Sigma V^T$ be the SVD of $A$, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

(a) Show that, for every $\epsilon > 0$, there is a full rank matrix $A_\epsilon \in \mathbb{R}^{n \times m}$ such that $\|A - A_\epsilon\|_2 = \epsilon$.

(b) Let $A_k = U \Sigma_k V^T$ where $\Sigma_k = \text{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ and $1 \leq k \leq r - 1$. Show that rank($A_k$) = $k$ and

$$\sigma_{k+1} = \|A - A_k\|_2 = \min \left\{ \|A - B\|_2 \mid \text{rank}(B) \leq k \right\}$$

(c) Assume that $r = \min(m, n)$. Let $B \in \mathbb{R}^{n \times m}$ and assume that $\|A - B\|_2 < \sigma_r$. Show that rank($B$) = $r$. 
