S.-T. Yau College Student Mathematics Contests 2015

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Let $\phi \in C([a,b],R)$. Suppose for every function $h \in C^1([a,b],R), h(a) = h(b) = 0$, we have

$$\int_a^b \phi(x)h(x)dx = 0.$$

Prove that $\phi(x) = 0$.

2. Let f be a Lebesgue integrable function over $[a, b+\delta], \delta > 0$, prove that

$$\lim_{h \to 0+} \int_a^b |f(x+h) - f(x)|dx \to 0.$$

3. Let $L(q, q^{'}, t)$ be a function of $(q, q^{'}, t) \in TU \times R, U$ is an open domain in R^n. Let $\gamma : [a,b] \to U$ be a curve in U. Define a functional $S(\gamma) = \int_a^b L(\gamma(t), \gamma^{'}(t), t)dt$. We say that γ is an extremal if for every smooth variation of $\gamma, \phi(t,s), s \in (-\delta, \delta), \phi(t,0) = \gamma(t), \phi_s = \phi(t,s)$, we have $\frac{dS(\phi_s)}{ds}|_{s=0} = 0$. Prove that every extremal γ satisfies the Euler-Lagrange equation: $\frac{d}{dt}(\frac{\partial L}{\partial q^{'}}) = \frac{\partial L}{\partial q}$.

4. Let $f : U \to U$ be a holomorphic function with U a bounded domain in the complex plane. Assuming $0 \in U, f(0) = 0, f^{'}(0) = 1$, prove that $f(z) = z$.

5. Let $T : H_1 \to H_2$ be a bounded operator of Hilbert spaces H_1, H_2. Let $S : H_1 \to H_2$ be a compact operator, that is, for every bounded sequence $\{v_n\} \in H_1, Sv_n$ has a converging subsequence. Show that $Coker(T+S) = H_2/\overline{Im(T+S)}$ is finite dimensional and $Im(T+S)$ is closed in H_2. (Hint: Consider equivalent statements in terms of adjoint operators.)

6. Let $u \in C^2(\bar{\Omega}), \Omega \subset R^d$ is a bounded domain with a smooth boundary.

1) Let u be a solution of the equation $\Delta u = f, u|_{\partial\Omega} = 0, f \in L^2(\Omega)$. Prove that there is a constant C depends only Ω such that

$$\int_\Omega (\Sigma_{j=1}^n (\frac{\partial u}{\partial x_j})^2 + u^2)dx \leq C \int_\Omega f^2(x)dx.$$

2) Let $\{u_n\}$ be a sequence of harmonic functions on Ω, such that $||u_n||_{L^2(\Omega)} \leq M < \infty$, for a constant M independent of n. Prove that there is a converging subsequence $\{u_{n_k}\}$ in $L^2(\Omega)$.

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Probability and Statistics

Team (5 problems)

Problem 1. One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

Problem 2. Assume a sequence of random variables X_n converges in distribution to a random variable X. Let $\{N_t, t \geq 0\}$ be a set of positive integer-valued random variables, which is independent of (X_n) and converges in probability to ∞ as $t \to \infty$. Prove that X_{N_t} converges in distribution to X as $t \to \infty$.

Problem 3. Suppose $T_1, T_2, \ldots, T_n$ is a sequence of independent, identically distributed random variables with the exponential distribution of the density function

$$p(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

Let $S_n = T_1 + T_2 + \cdots + T_n$. Find the distribution of the random vector

$$V_n = \left\{ \frac{T_1}{S_n}, \frac{T_2}{S_n}, \cdots, \frac{T_n}{S_n} \right\}.$$

Problem 4. Suppose that X and Z are jointly normal with mean zero and standard deviation 1. For a strictly monotonic function $f(\cdot)$, $\operatorname{cov}(X, Z) = 0$ if and only if $\operatorname{cov}(X, f(Z)) = 0$, provided the latter covariance exists. **Hint:** Z can be expressed as $Z = \rho X + \varepsilon$ where X and ε are independent and $\varepsilon \sim N(0, \sqrt{1-\rho^2})$.

Problem 5. Consider the following penalized least-squares problem (Lasso):

$$\frac{1}{2}\|\mathbf{Y} - \mathbf{X}\beta\|^2 + \lambda\|\beta\|_1$$

Let $\widehat{\beta}$ be a minimizer and $\mathbf{\Delta} = \widehat{\beta} - \beta^*$ for any given β^*. If $\lambda > 2\|\mathbf{X}^T(\mathbf{Y} - \mathbf{X}\beta^*)\|_\infty$, show that

1. $\|\mathbf{Y} - \mathbf{X}^T\widehat{\beta}\|^2 - \|\mathbf{Y} - \mathbf{X}^T\beta^*\|^2 > -\lambda\|\mathbf{\Delta}\|_1$.

2. $\|\mathbf{\Delta}_{S^c}\|_1 \leq 3\|\mathbf{\Delta}_S\|_1$, where $S = \{j : \beta_j^* \neq 0\}$ is the support of the vector β^*, S^c is its complement set, $\mathbf{\Delta}_S$ is the subvector of $\mathbf{\Delta}$ restricted on the set S, and $\|\mathbf{\Delta}_S\|_1$ is its L_1-norm.

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Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let $SO(3)$ be the set of all 3×3 real matrices A with determinant 1 and satisfying ${}^tAA = I$, where I is the identity matrix and ${}^tA$ is the transpose of A. Show that $SO(3)$ is a smooth manifold, and find its fundamental group. You need to prove your claims.

2. Let X be a topological space. The *suspension* $S(X)$ of X is the space obtained from $X \times [0, 1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

3. Let $F : M \to N$ be a smooth map between two manifolds. Let X_1, X_2 be smooth vector fields on M and let Y_1, Y_2 be smooth vector fields on N. Prove that if $Y_1 = F_*X_1$ and $Y_2 = F_*X_2$, then $F_*[X_1, X_2] = [Y_1, Y_2]$, where $[\ ,\ ]$ is the Lie bracket.

4. Let M_1 and M_2 be two compact convex closed surfaces in $\mathbb{R}^3$, and $f : M_1 \to M_2$ a diffeomerphism such that M_1 and M_2 have the same inner normal vectors and Gauss curvatures at the corresponding points. Prove that f is a translation.

5. Prove the second Bianchi identity:

$$R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0$$

6. Let M_1, M_2 be two complete n-dimensional Riemannian manifolds and $\gamma_i : [0, a] \to M_i$ are two arc length parametrized geodesics. Let ρ_i be the distance function to $\gamma_i(0)$ on M_i. Assume that $\gamma_i(a)$ is within the cut locus of $\gamma_i(0)$ and for any $0 \le t \le a$ we have the inequality of sectional curvatures

$$K_1(X_1, \frac{\partial}{\partial \gamma_1}) \ge K_2(X_2, \frac{\partial}{\partial \gamma_2}),$$

where $X_i \in T_{\gamma_i(t)}M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}$.

Then

$$Hess(\rho_1)(\widetilde{X}_1, \widetilde{X}_1) \le Hess(\rho_2)(\widetilde{X}_2, \widetilde{X}_2),$$

where $\widetilde{X}_i \in T_{\gamma_i(a)}M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}(a)$.

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Algebra and Number Theory

Team

This exam contains 6 problems. Please choose 5 of them to work on.

Problem 1. (20pt) Let $V = \mathbb{R}^n$ be an Euclidean space equipped with usual inner product, and g an orthogonal matrix acting on V. For $a \in V$, let s_a denote the reflection

$$s_a(x) := x - 2\frac{(x,a)}{(a,a)}a, \qquad \forall x \in V.$$

(1.1) (10pt) For $a = (g-1)b \neq 0$, show that

$$\ker(s_a g - 1) = \ker(g-1) \oplus \mathbb{R}b.$$

(1.2) (10pt) Show that g is a product of $\dim[(g-1)V]$ reflections.

Problem 2. (20pt) Let p and q be two distinct prime numbers. Let G be a non-abelian finite group satisfying the following conditions:

1. all nontrivial elements have order either p or q;
2. The q-Sylow subgroup H_q is normal and is a nontrivial abelian group.

Show in steps the following statement:

The group G is of the form $(\mathbb{Z}/p\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^n$, where the action of $1 \in \mathbb{Z}/p\mathbb{Z}$ on $(\mathbb{Z}/q\mathbb{Z})^n \simeq \mathbb{F}_q^n$ is given by a matrix $M(1) \in \mathrm{GL}_n(\mathbb{F}_q)$ whose eigenvalues are all primitive *p-th roots of unities.*

(2.1) (5pt) Let H_p denote a p-Sylow subgroup of G. Show that its inclusion into G induces an isomorphism $H_p \cong G/H_q$, and that $G \simeq H_p \ltimes H_q$.

(2.2) (5pt) Let $M : H_p \longrightarrow \mathrm{Aut}(H_q) \simeq \mathrm{GL}_n(\mathbb{F}_q)$ be the homomorphism induced from the conjugations. Show that for each $1 \neq a \in H_p$, $M(a)$ is semisimple whose eigenvalues are all *primitive* p-th roots of unities. In particular M is injective.

(2.3) (5pt) Show that if two nontrivial elements $a, b \in H_p$ commute with each other, then $a = b^n$ for some $n \in \mathbb{N}$, and that $H_p \simeq \mathbb{Z}/p\mathbb{Z}$.

(2.4) (5pt) Complete the solution of the problem.

Problem 3. (20pt) Let ζ be a root of unity satisfying an equation $\zeta = 1 + N\eta$ for an integer $N \geq 3$ and an algebraic integer η. Show that $\zeta = 1$.

Problem 4. (20pt) Let G be a finite group and (π, V) a finite dimensional $\mathbb{C}G$-module. For $n \geq 0$, let $\mathbb{C}[V]_n$ be the space of homogeneous polynomial functions on V of degree n. For a simple G-representation ρ, denote by $a_n(\rho)$ the multiplicity of ρ in $\mathbb{C}[V]_n$. Show that

$$\sum_{n\geq 0} a_n(\rho)t^n = \frac{1}{|G|}\sum_{g\in G}\frac{\overline{\chi_\rho(g)}}{\det(\mathrm{id}_V - \pi(g)t)}.$$

Problem 5. (20pt) Let A be an $n \times n$ complex matrix considered as an operator on $V = (\mathbb{C}^n, (\cdot,\cdot))$ with standard hermitian form. Let $A^* = \bar{A}^t$ be the hermitian transpose of A:

$$(Ax, y) = (x, A^*y), \qquad \forall x, y \in \mathbb{C}^n.$$

(5.1) (5pt) For any $\lambda \in \mathbb{C}$, show the identity:

$$\ker(A-\lambda)^\perp = (A^* - \bar{\lambda})V.$$

(5.2) (15pt) Show the equivalence of the following two statements:

(a) A commutes with A^*;

(b) there is a unitary matrix U (in the sense $U^* = U^{-1}$), such that UAU^{-1} is diagonal.

Problem 6. (20pt) Consider the polynomial $f(x) = x^5 - 80x + 5$.

(6.1) (5pt) Show that f is irreducible over $\mathbb{Q}$;

(6.2) (15 pt) Show in steps that the split field K of f has Galois group $G := \mathrm{Gal}(K/\mathbb{Q})$ isomorphic to S_5, the symmetric group of 5 letters.

(a) (5pt) $f = 0$ has exactly two complex roots;

(b) (5pt) G can be embedded into S_5 with image containing cycles (12345) and (12);

(c) (5pt) $G \simeq S_5$.

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Applied Math. and Computational Math.

Team (5 problems)

Problem 1. Consider the elliptic interface problem

$$(a(x)u_x)_x = f,\ x \in (0,1)$$

with the Dirichlet boundary condition

$$u(0) = u(1) = 0.$$

Here, f is a smooth function, the elliptic coefficient $a(x)$ is discontinuous across an interface point ξ, that is,

$$a(x) = \begin{cases} a_0 & \text{for } 0 < x < \xi \\ a_1 & \text{for } \xi < x < 1, \end{cases}$$

$a_0, a_1 > 0$ are positive constants, and $0 < \xi < 1$ is an interface point. Across the interface, we need to impose two jump conditions

$$u(\xi-) = u(\xi+),\ a(\xi-)u_x(\xi-) = a(\xi+)u_x(\xi+).$$

Question:

1. (25%) Design a numerical method to solve this problem. The method should be at least first order. It is better to be high order (if your method is first order, you get 20% points).
2. (75%) Prove your accuracy and convergence arguments (if your method is first order, you get 60% points).

Problem 2. Let G be graph of a social network, where for each pair of members there is either no connection, or a positive or a negative one.

An unbalanced cycle in G is a a cycle which have odd number of negative edges. Traversing along such a cycle with social rules such as friend of enemy are enemy would result in having a negative relation of one with himself!

A resigning in G at a vertex v of G is to switch the type (positive or negative) of all edges incident to v.

Question: Show that one can switch all edge of G into positive edges using a sequence resigning if and only if there is no unbalanced cycle in G.

Problem 3. We consider particles which are able to produce new particles of like kind. A single particle forms the original, or zero, generation. Every particle has probability p_k ($k = 0, 1, 2, \dots$) of creating exactly k new particles; the direct descendants of the nth generation form the $(n + 1)$st generation. The particles of each generation act independently of each other.

Assume $0 < p_0 < 1$. Let $P(x) = \sum_{k\geq 0} p_k x^k$ and $\mu = P'(1) = \sum_{k\geq 0} kp_k$ be the expected number of direct descendants of one particle. Prove that if $\mu > 1$, then the probability x_n that the process terminates at or before the nth generation tends to the unique root $\sigma \in (0,1)$ of equation $\sigma = P(\sigma)$.

Problem 4. (Isopermetric inequality). Consider a closed plane curve described by a parametric equation $(x(t), y(t)), 0 \leq t \leq T$ with parameter t oriented counterclockwise and $(x(0), y(0)) = (x(T), y(T))$.

(a): Show that the total length of the curve is given by

$$L = \int_0^T \sqrt{(x'(t))^2 + (y'(t))^2)}\, dt$$

(b): Show that the total area enclosed by the curve is given by

$$A = \frac{1}{2} \int_0^T \big(x(t)y'(t) - y(t)x'(t)\big)\, dt$$

(c): The classical iso-perimetric inequality states that for closed plane curves with a fixed length L, circles have the largest enclosed area A. Formulate this question into a variational problem.

(d): Derive the Euler-Lagrange equation for the variational problem in **(c)**.

(e): Show that there are two constants x_0 and y_0 such that

$$(x(t) - x_0)^2 + (y(t) - y_0)^2 \equiv r^2$$

where $r = L/(2\pi)$. Explain your result.

Problem 5. Let $A \in \mathbb{R}^{n\times m}$ with rank $r < \min(m, n)$. Let $A = U\Sigma V^T$ be the SVD of A, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

(a) Show that, for every $\epsilon > 0$, there is a full rank matrix $A_\epsilon \in \mathbb{R}^{n\times m}$ such that $||A - A_\epsilon||_2 = \epsilon$.

(b) Let $A_k = U\Sigma_k V^T$ where $\Sigma_k = \mathrm{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ and $1 \leq k \leq r-1$. Show that $\mathrm{rank}(A_k) = k$ and

$$\sigma_{k+1} = ||A - A_k||_2 = \min\left\{||A - B||_2 \quad | \quad \mathrm{rank}(B) \leq k\right\}$$

(c) Assume that $r = \min(m, n)$. Let $B \in \mathbb{R}^{n\times m}$ and assume that $||A - B||_2 < \sigma_r$. Show that $\mathrm{rank}(B) = r$.