S.-T. Yau College Student Mathematics Contests 2015

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Let $f_n \in L^2(R)$ be a sequence of measurable functions over the line, $f_n \to f$ almost everywhere. Let $||f_n||_{L^2} \to ||f||_{L^2}$, prove that $||f_n - f||_{L^2} \to 0$.

2. Let f be a continuous function on $[a, b]$, define $M_n = \int_a^b f(x)x^n dx$. Suppose that $M_n = 0$ for all integers $n \geq 0$, show that $f(x) = 0$ for all x.

3. Determine all entire functions f that satisfying the inequality

$$|f(z)| \leq |z|^2 |Im(z)|^2$$

for z sufficiently large.

4. Describe all functions that are holomorphic over the unit disk $D = \{z||z| < 1\}$, continuous on $\bar{D}$ and map the boundary of the disk into the boundary of the disk.

5. Let $T : H_1 \to H_2, Q : H_2 \to H_1$ be bounded linear operators of Hilbert spaces H_1, H_2. Let $QT = Id - S_1, TQ = Id - S_2$ where S_1 and S_2 are compact operators. Prove $KerT = \{v \in H_1, Tv = 0\}, CokerT = H_2/\overline{Im(T)}$ are finite dimensional and $Im(T) = \{Tv \in H_2, v \in H_1\}$ is closed in H_2.

Note: S is compact means for every bounded sequence $x_n \in H_1, Sx_n$ has a converging subsequence.

6. Let H_1 be the Sobolev space on the unit interval $[0, 1]$, i.e. the Hilbert space consisting of functions $f \in L^2([0, 1])$ such that

$$||f||_1^2 = \Sigma_{n=-\infty}^{\infty} (1 + n^2)|\hat{f}(n)|^2 < \infty;$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^1 f(x) e^{-2\pi i n x} dx$$

are Fourier coefficients of f. Show that there exists constant $C > 0$ such that

$$||f||_{L^\infty} \leq C||f||_1$$

for all $f \in H_1$, where $||.||_{L^\infty}$ stands for the usual supremum norm. (Hint: Use Fourier series.)