S.-T. Yau College Student Mathematics Contests 2015

Applied Math. and Computational Math.

Individual (5 problems)

Problem 1. Let r and s be relatively prime positive integers. Prove that the number of lattice paths from $(0,0)$ to (r,s), which consists of steps $(1,0)$ and $(0,1)$ and never go above the line $ry = sx$ is given by

$$\frac{1}{r+s}\binom{r+s}{s}.$$

Problem 2. The following 2×2 block matrix

$$C(\alpha) = \begin{bmatrix} \alpha I & A \\ A^T & 0 \end{bmatrix}$$

plays a key role in an augmented system method to solve linear least squares problem, a fundamental numerical linear algebra problem for fitting a linear model to observations subject to errors in science, where $A \in \mathbf{R}^{m\times n}$ is of full rank $n \le m$, I is a $m\times m$ identity matrix, and $\alpha \ge 0$. Prove the following results which address the question of optimal choice of scaling α for stabiltiy of the augmented system method.

(a) The eigenvalues of $C(\alpha)$ are

$$\frac{\alpha}{2} \pm \left(\frac{\alpha^2}{4} + \sigma_i^2\right)^{1/2} \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and} \quad \alpha \quad (m-n \text{ times}),$$

where σ_i for $i = 1, 2, \ldots, n$ are the singular values of A, arranged in the decreasing order, i.e., $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$.

(b) The condition number $\kappa_2(C(\alpha)) = \|C(\alpha)\|_2 \, \|[C(\alpha)]^{-1}\|_2$ has the following bounds:

$$\sqrt{2}\kappa_2(A) \le \min_\alpha \kappa_2(C(\alpha)) \le 2\kappa_2(A),$$

with $\min_\alpha \kappa_2(C(\alpha))$ being achieved for $\alpha = \sigma_n/\sqrt{2}$, and

$$\max_\alpha \kappa_2(C(\alpha)) > \kappa_2(A)^2,$$

where $\|\cdot\|$ is the spectral norm of a matrix.

Recall that any matrix $A \in \mathbf{R}^{m\times n}$ has a singular value decomposition (SVD):

$$A = U\Sigma V^T, \quad \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p) \in \mathbf{R}^{m\times n}, \quad p = \min(m,n),$$

where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_p \ge 0$, and $U \in \mathbf{R}^{m\times m}$, $V \in \mathbf{R}^{n\times n}$ are both orthogonal. The σ_i are the singular values of A and the columns of U and V are the left and right singular vectors of A, respectively.

Problem 3. Solve the following linear hyperbolic partial differential equation

$$u_t + au_x = 0, \quad t \geq 0, \tag{1}$$

where a is a constant. Using the finite difference approximation, we can obtain the forward-time central-space scheme as follows,

$$\frac{u_m^{n+1} - u_m^n}{k} + a\frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0, \tag{2}$$

where k and h are temporal and spatial mesh sizes.

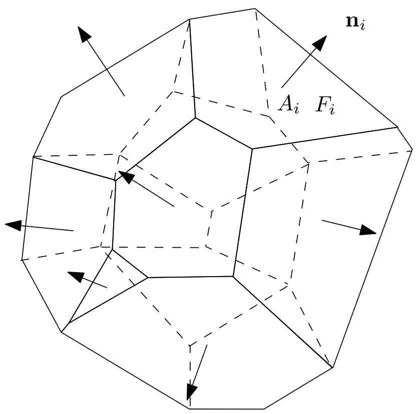
(a) Show that when we fix $\lambda = k/h$ as a positive constant, the forward-time central-space scheme (2) is consistent with equation (1).
(b) Analyze the stability of this method. Is the method stable with $\lambda = k/h$ being fixed as a constant?
(c) How would the answer change if you are allowed to make $\lambda = k/h$ small?
(d) Would this is a good scheme to use even if you can make it stable by making λ small? If not, please provide a simple modification to make this scheme stable by keeping λ fixed.

Problem 4. Let $A, H, Q \in \mathbb{C}^{n\times n}$ and Q is non-singular. Assume that $H = Q^{-1}AQ$ and H is properly upper Hessenberg. Show that

$$\text{span}\{q_1, q_2, \ldots, q_j\} = \mathcal{K}_j(A, q_1), \qquad j = 1, 2, \ldots, n$$

where q_j is the j-th column of Q, and $\mathcal{K}_j(A, q_1) = \text{span}\{q_1, Aq_1, \ldots, A^{j-1}q_1\}$.

Problem 5. Minkowski Problem.



Assume P is a convex polyhedron embedded in $\mathbb{R}^3$, the faces are $\{F_1, F_2, \cdots, F_k\}$, the unit normal vector to the face F_i is $\mathbf{n}_i$, the area of F_i is A_i, $1 \leq i \leq k$.

- Show that

$$A_1\mathbf{n}_1 + A_2\mathbf{n}_2 + \cdots A_k\mathbf{n}_k = \mathbf{0}, \tag{3}$$

- Given k unit vectors $\{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_k\}$ which can not be contained in any half space, and k real positive numbers $\{A_1, A_2, \cdots, A_k\}$, $A_i > 0$, and satisfying the condition (3), show that there exists a convex polyhedron P, whose face normals are $\mathbf{n}_i$'s, face areas are A_i's.