S.-T. Yau College Student Mathematics Contests 2015

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let n, m be positive integers. Show that the product of spheres $S^n \times S^m$ has trivial tangent bundle if and only if n or m is odd.

2. Show that there does not exist a compact three-dimensional manifold M whose boundary is the real projective space $\mathbb{RP}^2$.

3. Let M^n be a smooth manifold without boundary and X a smooth vector field on M. If X does not vanish at $p \in M$, show that there exists a local coordinate chart $(U; x_1, \ldots, x_n)$ centered at p such that in U the vector field X takes the form $X = \frac{\partial}{\partial x_1}$.

4. Let $M \to \mathbb{R}^3$ be a compact simply-connected closed surface. Prove that if M has constant mean curvature, then M is a standard sphere.

5. Let M be an n-dimensional compact Riemannian manifold with diameter π/c and Ricci curvature $\geq (n-1)c^2 > 0$. Show that M is isometric to the standard n-sphere in $\mathbb{R}^{n+1}$ with radius $1/c$.

6. Suppose (M, g) is a Riemannian manifold and $p \in M$. Show that the second-order Taylor series of g in normal coordinates centered at p is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{k,l} R_{iklj} x_k x_l + O(|x|^3).$$