S.-T. Yau College Student Mathematics Contests 2015

Probability and Statistics

Individual (5 problems)

Problem 1. (a) Let X and Y be two random variables with zero means, variance 1, and correlation ρ. Prove that

$$\mathbb{E}[\max\{X^2, Y^2\}] \leq 1 + \sqrt{1-\rho^2}.$$

(b) Let X and Y have a bivariate normal distribution with zero means, variances σ^2 and τ^2, respectively, and correlation ρ. Find the conditional expectation $\mathbb{E}(X|Y)$.

Problem 2. We flip a fair coin until heads turns out twice consecutively. What is the expected number of flips?

Problem 3. Let $(X_n, n \geq 1)$ be a sequence of independent Gaussian variables, with respective mean μ_n, and variance σ_n^2.

(a) Prove that if $\sum_n X_n^2$ converges in L^1, then $\sum_n X_n^2$ converges in L^p, for every $p \in [1, \infty)$.

(b) Assume that $\mu_n = 0$, for every n. Prove that if $\sum_n \sigma_n^2 = \infty$, then

$$\mathbb{P}(\sum_n X_n^2 = \infty) = 1.$$

Problem 4. Let $X_1, \ldots, X_n$ be a random sample of size n from the exponential distribution with pdf $f(x;\theta) = \theta^{-1}\exp(-x/\theta)$ for $x, \theta > 0$, zero elsewhere. Let $Y_1 = \min\{X_1, \ldots, X_n\}$. Consider an estimator nY_1.

(a) Show this estimate is unbiased.

(b) Prove or disprove: This estimate is a consistent estimator.

(c) Prove or disprove: This estimate is an efficient estimator.

Problem 5. Let the independent normal random variables $Y_1, \ldots, Y_n$ have, respectively, the probability density functions $N(\mu, \gamma^2 x_i^2)$, $i = 1, \ldots, n$, where the given $x_1, \ldots, x_n$ are not all equal and no one of which is zero.

(a) Construct a confidence interval for γ with significance level $1-\alpha$.

(b) Discuss the test of the hypothesis $H_0 : \gamma = 1, \mu$ unspecified, against all alternatives $H_1 : \gamma \neq 1, \mu$ unspecified.