The 8^{th} Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Let x and y be positive real numbers such that $x+y^{2016} \geq 1$. Prove that $x^{2016}+y > 1-1/100$.

Solution. If $x \geq 1-1/(100\cdot 2016)$, then

$$x^{2016} \geq \left(1-\frac{1}{100\cdot 2016}\right)^{2016} > 1-2016\cdot\frac{1}{100\cdot 2016} = 1-\frac{1}{100}$$

by Bernoulli's inequality, whence the conclusion.

If $x < 1-1/(100\cdot 2016)$, then $y \geq (1-x)^{1/2016} > (100\cdot 2016)^{-1/2016}$, and it is sufficient to show that the latter is greater than $1-1/100 = 99/100$; alternatively, but equivalently, that

$$\left(1+\frac{1}{99}\right)^{2016} > 100\cdot 2016.$$

To establish the latter, refer again to Bernoulli's inequality to write

$$\left(1+\frac{1}{99}\right)^{2016} > \left(1+\frac{1}{99}\right)^{99\cdot 20} > \left(1+99\cdot\frac{1}{99}\right)^{20} = 2^{20} > 100\cdot 2016.$$

Remarks. (1) Although the constant $1/100$ is not sharp, it cannot be replaced by the smaller constant $1/400$, as the values $x = 1-1/210$ and $y = 1-1/380$ show.

(2) It is natural to ask whether $x^n+y \geq 1-1/k$, whenever x and y are positive real numbers such that $x+y^n \geq 1$, and k and n are large. Using the inequality $\left(1+\frac{1}{k-1}\right)^k > \mathrm{e}$, it can be shown along the lines in the solution that this is indeed the case if $k \leq \frac{n}{2\log n}(1+o(1))$. It *seems* that this estimate differs from the best one by a constant factor.

Problem 5. A convex hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a circle Ω of radius R. The diagonals A_1B_2, A_2B_3, and A_3B_1 concur at X. For $i = 1, 2, 3$, let ω_i be the circle tangent to the segments XA_i and XB_i, and to the arc A_iB_i of Ω not containing other vertices of the hexagon; let r_i be the radius of ω_i.

(a) Prove that $R \geq r_1 + r_2 + r_3$.

(b) If $R = r_1 + r_2 + r_3$, prove that the six points where the circles ω_i touch the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic.

Solution. (a) Let ℓ_1 be the tangent to Ω parallel to A_2B_3, lying on the same side of A_2B_3 as ω_1. The tangents ℓ_2 and ℓ_3 are defined similarly. The lines ℓ_1 and ℓ_2, ℓ_2 and ℓ_3, ℓ_3 and ℓ_1 meet at C_3, C_1, C_2, respectively (see Fig. 1). Finally, the line C_2C_3 meets the rays XA_1 and XB_1 emanating from X at S_1 and T_1, respectively; the points S_2, T_2, and S_3, T_3 are defined similarly.

Each of the triangles $\Delta_1 = \triangle XS_1T_1$, $\Delta_2 = \triangle T_2XS_2$, and $\Delta_3 = \triangle S_3T_3X$ is similar to $\Delta = \triangle C_1C_2C_3$, since their corresponding sides are parallel. Let k_i be the ratio of similitude of Δ_i and Δ (e.g., $k_1 = XS_1/C_1C_2$ and the like). Since $S_1X = C_2T_3$ and $XT_2 = S_3C_1$, it follows that $k_1+k_2+k_3 = 1$, so, if ρ_i is the inradius of Δ_i, then $\rho_1+\rho_2+\rho_3 = R$.

Finally, notice that ω_i is interior to Δ_i, so $r_i \leq \rho_i$, and the conclusion follows by the preceding.

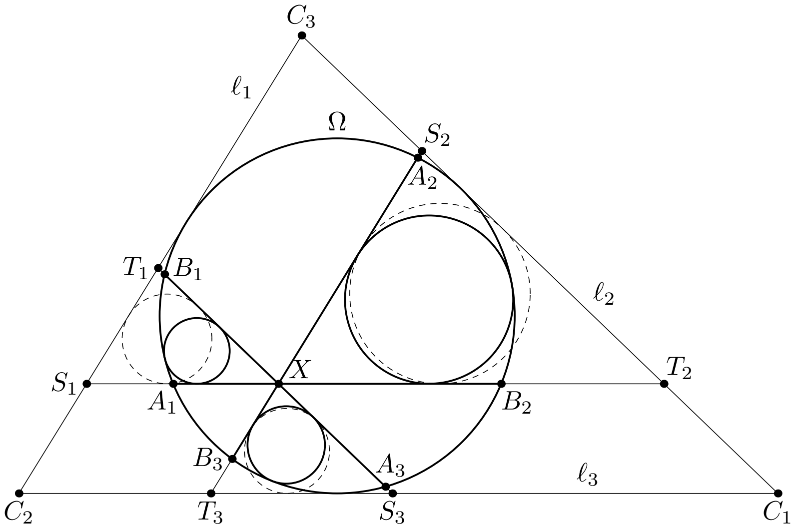

Fig. 1

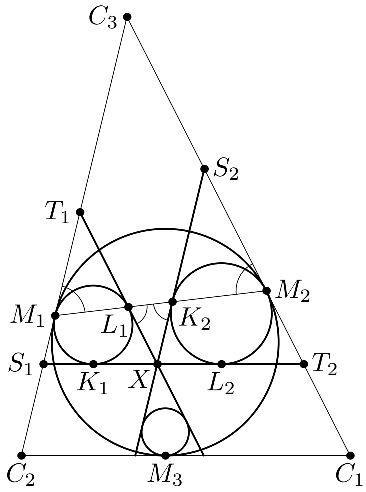

Fig. 2

(b) By part **(a)**, the equality $R = r_1 + r_2 + r_3$ holds if and only if $r_i = \rho_i$ for all i, which implies in turn that ω_i is the incircle of Δ_i. Let K_i, L_i, M_i be the points where ω_i touches the sides XS_i, XT_i, S_iT_i, respectively. We claim that the six points K_i and L_i ($i = 1, 2, 3$) are equidistant from X.

Clearly, $XK_i = XL_i$, and we are to prove that $XK_2 = XL_1$ and $XK_3 = XL_2$. By similarity, $\angle T_1M_1L_1 = \angle C_3M_1M_2$ and $\angle S_2M_2K_2 = \angle C_3M_2M_1$, so the points M_1, M_2, L_1, K_2 are collinear. Consequently, $\angle XK_2L_1 = \angle C_3M_1M_2 = \angle C_3M_2M_1 = \angle XL_1K_2$, so $XK_2 = XL_1$. Similarly, $XK_3 = XL_2$.

Remark. Under the assumption in part **(b)**, the point M_i is the centre of a homothety mapping ω_i to Ω. Since this homothety maps X to C_i, the points M_i, C_i, X are collinear, so X is the *Gergonne point* of the triangle $C_1C_2C_3$. This condition is in fact equivalent to $R = r_1 + r_2 + r_3$.

Problem 6. A set of n points in Euclidean 3-dimensional space, no four of which are coplanar, is partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$. An $\mathcal{AB}$-*tree* is a configuration of $n - 1$ segments, each of which has an endpoint in $\mathcal{A}$ and the other in $\mathcal{B}$, and such that no segments form a closed polyline. An $\mathcal{AB}$-tree is transformed into another as follows: choose three distinct segments A_1B_1, B_1A_2 and A_2B_2 in the $\mathcal{AB}$-tree such that A_1 is in $\mathcal{A}$ and $A_1B_1 + A_2B_2 > A_1B_2 + A_2B_1$, and remove the segment A_1B_1 to replace it by the segment A_1B_2. Given any $\mathcal{AB}$-tree, prove that every sequence of successive transformations comes to an end (no further transformation is possible) after finitely many steps.

Solution. The configurations of segments under consideration are all bipartite geometric trees on the points n whose vertex-parts are $\mathcal{A}$ and $\mathcal{B}$, and transforming one into another preserves the degree of any vertex in $\mathcal{A}$, but not necessarily that of a vertex in $\mathcal{B}$.

The idea is to devise a strict semi-invariant of the process, i.e., assign each $\mathcal{AB}$-tree a real number strictly decreasing under a transformation. Since the number of trees on the n points is finite, the conclusion follows.

To describe the assignment, consider an $\mathcal{AB}$-tree $\mathcal{T} = (\mathcal{A} \sqcup \mathcal{B}, \mathcal{E})$. Removal of an edge e of $\mathcal{T}$ splits the graph into exactly two components. Let $p_{\mathcal{T}}(e)$ be the number of vertices in $\mathcal{A}$ lying in the component of $\mathcal{T} - e$ containing the $\mathcal{A}$-endpoint of e; since $\mathcal{T}$ is a tree, $p_{\mathcal{T}}(e)$ counts the number of paths in $\mathcal{T} - e$ from the $\mathcal{A}$-endpoint of e to vertices in $\mathcal{A}$ (including the one-vertex path). Define $f(\mathcal{T}) = \sum_{e \in \mathcal{E}} p_{\mathcal{T}}(e)|e|$, where $|e|$ is the Euclidean length of e.

We claim that f strictly decreases under a transformation. To prove this, let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by a transformation involving the polyline $A_1B_1A_2B_2$; that is, A_1 and A_2 are in $\mathcal{A}$, B_1

and B_2 are in $\mathcal{B}$, $A_1B_1 + A_2B_2 > A_1B_2 + A_2B_1$, and $\mathcal{T}' = \mathcal{T} - A_1B_1 + A_1B_2$. It is readily checked that $p_{\mathcal{T}'}(e) = p_{\mathcal{T}}(e)$ for every edge e of $\mathcal{T}$ different from A_1B_1, A_2B_1 and A_2B_2, $p_{\mathcal{T}'}(A_1B_2) = p_{\mathcal{T}}(A_1B_1)$, $p_{\mathcal{T}'}(A_2B_1) = p_{\mathcal{T}}(A_2B_1) + p_{\mathcal{T}}(A_1B_1)$, and $p_{\mathcal{T}'}(A_2Bb_2) = p_{\mathcal{T}}(A_2B_2) - p_{\mathcal{T}}(A_1B_1)$. Consequently,

$$\begin{aligned} f(\mathcal{T}') - f(\mathcal{T}) &= p_{\mathcal{T}'}(A_1B_2) \cdot A_1B_2 + (p_{\mathcal{T}'}(A_2B_1) - p_{\mathcal{T}}(A_2B_1)) \cdot A_2B_1 + \\ &\quad (p_{\mathcal{T}'}(A_2B_2) - p_{\mathcal{T}}(A_2B_2)) \cdot A_2B_2 - p_{\mathcal{T}}(A_1B_1) \cdot A_1B_1 \\ &= p_{\mathcal{T}}(A_1B_1)\,(A_1B_2 + A_2B_1 - A_2B_2 - A_1B_1) < 0. \end{aligned}$$

Remarks. (1) The solution above does not involve the geometric structure of the configurations, so the conclusion still holds if the Euclidean length (distance) is replaced by any real-valued function on $\mathcal{A} \times \mathcal{B}$.

(2) There are infinitely many strict semi-invariants that can be used to establish the conclusion, as we are presently going to show. The idea is to devise a non-strict real-valued semi-invariant f_A for each A in $\mathcal{A}$ (i.e., f_A does not increase under a transformation) such that $\sum_{A\in\mathcal{A}} f_A = f$. It then follows that any linear combination of the f_A with positive coefficients is a strict semi-invariant.

To describe f_A, where A is a fixed vertex in $\mathcal{A}$, let $\mathcal{T}$ be an $\mathcal{AB}$-tree. Since $\mathcal{T}$ is a tree, by orienting all paths in $\mathcal{T}$ with an endpoint at A away from A, every edge of $\mathcal{T}$ comes out with a unique orientation so that the in-degree of every vertex of $\mathcal{T}$ other than A is 1. Define $f_A(\mathcal{T})$ to be the sum of the Euclidean lengths of all out-going edges from $\mathcal{A}$. It can be shown that f_A does not increase under a transformation, and it strictly decreases if the paths from A to each of A_1, A_2, B_1, B_2 all pass through A_1 — i.e., of these four vertices, A_1 is combinatorially nearest to A. In particular, this is the case if $A_1 = A$, i.e., the edge-switch in the transformation occurs at A. It is not hard to prove that $\sum_{A\in\mathcal{A}} f_A(\mathcal{T}) = f(\mathcal{T})$.

The conclusion of the problem can also be established by resorting to a single carefully chosen f_A. Suppose, if possible, that the process is infinite, so some tree $\mathcal{T}$ occurs (at least) twice. Let A be the vertex in $\mathcal{A}$ at which the edge-switch occurs in the transformation of the first occurrence of $\mathcal{T}$. By the preceding paragraph, consideration of f_A shows that $\mathcal{T}$ can never occur again.

(3) Recall that the degree of any vertex in $\mathcal{A}$ is invariant under a transformation, so the linear combination $\sum_{A\in\mathcal{A}}(\deg A - 1)f_A$ is a strict semi-invariant for $\mathcal{AB}$-trees $\mathcal{T}$ whose vertices in $\mathcal{A}$ all have degrees exceeding 1. Up to a factor, this semi-invariant can alternatively, but equivalently be described as follows. Fix a vertex $*$ and assign each vertex X a number $g(X)$ so that $g(*) = 0$, and $g(A) - g(B) = AB$ for every A in $\mathcal{A}$ and every B in $\mathcal{B}$ joined by an edge. Next, let $\beta(\mathcal{T}) = \frac{1}{|\mathcal{B}|}\sum_{B\in\mathcal{B}} g(B)$, let $\alpha(\mathcal{T}) = \frac{1}{|\mathcal{E}|-|\mathcal{A}|}\sum_{A\in\mathcal{A}}(\deg A - 1)g(A)$, where $\mathcal{E}$ is the edge-set of $\mathcal{T}$, and set $\mu(\mathcal{T}) = \beta(\mathcal{T}) - \alpha(\mathcal{T})$. It can be shown that μ strictly decreases under a transformation; in fact, μ and $\sum_{A\in\mathcal{A}}(\deg A - 1)f_A$ are proportional to one another.