S.-T. Yau College Student Mathematics Contests 2016

Algebra and Number Theory

Individual

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Let E be a linear space over $\mathbb{R}$, of finite dimension $n \geq 2$, equipped with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $u_1, u_2, \ldots, u_n$ be a basis of E. Let $v_1, v_2, \ldots, v_n$ be the dual basis, that is,

$$\langle u_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $i, j = 1, 2, \ldots, n$.

(a) (8 points) Assume that $\langle u_i, u_j \rangle \leq 0$ for all $1 \leq i < j \leq n$. Show that there is an orthogonal basis $u'_1, u'_2, \ldots, u'_n$ of E such that u'_i is a non-negative linear combination of $u_1, u_2, \ldots, u_i$, for all $i = 1, 2, \ldots, n$.

(b) (6 points) With the same assumption as in Part (a), show that $\langle v_i, v_j \rangle \geq 0$ for all $1 \leq i < j \leq n$.

(c) (6 points) Assume that $n \geq 3$. Show that the condition $\langle u_i, u_j \rangle \geq 0$ for all $1 \leq i < j \leq n$ does not imply that $\langle v_i, v_j \rangle \leq 0$ for all $1 \leq i < j \leq n$.

Problem 2 (20 points)**.** Let $d \geq 1$ and $n \geq 1$ be integers.

(a) (5 points) Show that there are only finitely many subgroups $G \subseteq \mathbb{Z}^d$ of index n. Let $f_d(n)$ denote the number of such subgroups.

(b) (5 points) Let $g_d(n)$ denote the number of subgroups $H \subseteq \mathbb{Z}^d$ of index n such that the quotient group is cyclic. Show that $f_d(mn) = f_d(m)f_d(n)$ and $g_d(mn) = g_d(m)g_d(n)$ for coprime positive integers m and n.

(c) (5 points) Compute $g_d(p^r)$ for every prime power p^r, $r \geq 1$.

(d) (5 points) Compute $f_2(20)$.

Problem 3 (20 points)**.** Let A be a complex $m \times m$ matrix. Assume that there exists an integer $N \geq 0$ such that $t_n = \mathrm{tr}(A^n)$ is an algebraic integer for all $n \geq N$. The goal of this problem is to show that the eigenvalues $a_1, \ldots, a_m$ of A are algebraic integers.

(a) (10 points) Show that there exist algebraic numbers $b_{ij} \in \mathbb{C}$, $1 \leq i, j \leq m$ such that

$$a_i^n = \sum_{j=1}^{m} b_{ij} t_{n+j-1}$$

for all $n \geq 0$ and all $1 \leq i \leq m$. In particular, $a_1, \ldots, a_m$ are algebraic numbers.

(b) (8 points) Let R be the ring of all algebraic integers in $\mathbb{C}$ and let K be the field of all algebraic numbers in $\mathbb{C}$. Show that for $a \in K$, if $R[a]$ is contained in a finitely-generated R-submodule of K, then $a \in R$.

(c) (2 points) Conclude that $a_1, \ldots, a_m$ are algebraic integers.

Problem 4 (20 points)**.** Let E be a Euclidean plane. For each line l in E, write $s_l \in \mathrm{Iso}(E)$ for the reflection with respect to l, where $\mathrm{Iso}(E)$ denotes the group of distance-preserving bijections from E to itself.

(a) (6 points) Let l_1 and l_2 be two distinct lines in E. Find the necessary and sufficient condition that s_{l_1} and s_{l_2} generate a finite group.

(b) (7 points) Let l_1, l_2 and l_3 be three pairwise distinct lines in E. Assume that s_{l_1}, s_{l_2} and s_{l_3} generate a finite group. Show that l_1, l_2, l_3 intersect at a point.

(c) (7 points) Let G be a finite subgroup of $\mathrm{Iso}(E)$ generated by reflections. Show that G is generated by at most two reflections.

Problem 5 (20 points)**.** Let G be a finite group of order $2^n m$ where $n \geq 1$ and m is an odd integer. Assume that G has an element of order 2^n. The goal of this problem is to show that G has a normal subgroup of order m.

(a) (5 points) Show that if M is a normal subgroup of G of order m, then M is the only subgroup of G of order m.

(b) (5 points) Let N be a normal subgroup of G and let P be a 2-Sylow subgroup of G. Show that $P \cap N$ is a 2-Sylow subgroup of N.

(c) (5 points) Show that the homomorphism $G \to \{\pm 1\}$ carrying g to $\mathrm{sgn}(l_g)$ is surjective. Here $\mathrm{sgn}(l_g)$ denotes the sign of the permutation $l_g \colon G \to G$ given by left multiplication by g.

(d) (5 points) Deduce by induction on n that G has a normal subgroup of order m.