S.-T. Yau College Student Mathematics Contests 2016

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Suppose that F is continuous on $[a,b], F^{'}(x)$ exists for every $x \in (a,b), F^{'}(x)$ is integrable. Prove that F is absolutely continuous and

$$F(b)-F(a)=\int_a^b F^{'}(x)dx.$$

2. Suppose that f is integrable on $\mathbf{R}^n$, let $K_\delta(x)=\delta^{-\frac{n}{2}}e^{\frac{-\pi|x|^2}{\delta}}$ for each $\delta>0$. Prove that the convolution

$$(f*K_\delta)(x)=\int_{\mathbf{R}^n} f(x-y)K_\delta(y)dy$$

is integrable and $||(f*K_\delta)-f||_{L^1(\mathbf{R}^n)}\to 0$, as $\delta\to 0$.

3. Prove that a bounded function on interval $I=[a,b]$ is Riemann integrable if and only if its set of discontinuities has measure zero. You may prove this by the following steps.

Define $I(c,r)=(c-r,c+r), osc(f,c,r)=\sup_{x,y\in J\cap I(c,r)}|f(x)-f(y)|, osc(f,c)=\lim_{r\to 0} osc(f,r,c)$.

1) f is continuous at $c\in J$ if and only if $osc(f,c)=0$.

2) For arbitrary $\epsilon>0, \{c\in J|osc(f,c)\geq\epsilon\}$ is compact.

3) If the set of discontinuities of f has measure 0, then f is Riemann integrable.

4. 1) Let f be the Rukowski map: $w=\frac{1}{2}(z+\frac{1}{z})$. Show that it maps $\{z\in\bar{\mathbf{C}}||z|>1\}$ to $\bar{\mathbf{C}}/[-1,1], \bar{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$.

2) Compute the integral:

$$\int_0^\infty \frac{\log x}{x^2-1}dx.$$

5. Let f be a doubly periodic meromorphic function over the complex plane, i.e. $f(z+1)=f(z), f(z+i)=f(z), z\in\mathbf{C}$, prove that the number of zeros and the number of poles are equal.

6. Let A be a bounded self-adjoint operator over a complex Hilbert space. Prove that the spectrum of A is a bounded closed subset of the real line $\mathbf{R}$.