S.-T. Yau College Student Mathematics Contests 2016

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let M be a compact odd-dimensional manifold with boundary ∂M. Show that the Euler characteristics of M and ∂M are related by:

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

2. Compute the de Rham cohomology of a punctured two-dimensional torus $T^2 - \{p\}$, where $p \in T^2$. If $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates $(x, y) \in \mathbb{R}^2$, then is the volume form $\omega = dx \wedge dy$ exact?

3. Let $M^n \to \mathbb{R}^{n+1}$ be a closed oriented hypersurface. The r-th mean curvature of M^n is defined by

$$H_r := \frac{1}{\binom{n}{r}} \sum_{i_1 < i_2 \cdots < i_r} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}, \quad (1 \le r \le n)$$

where λ_i $(i = 1, \cdots, n)$ are principal curvatures of M^n. Prove that if all of λ_i are positive and H_r =constant for a certain r, then M^n is a hypersphere in $\mathbb{R}^{n+1}$.

4. State and prove the cut-off function lemma on a differentiable manifold.

5. Let M be a compact Riemannian manifold without boundary. Show that if M has positive Ricci curvature, then $H^1(M, \mathbb{R}) = 0$.

6. Let M be an orientable, closed and embedded minimal hypersurface in S^{n+1}. Denote by λ_1 the first eigenvalue for the Laplace-Beltrami operator on M. Prove that $\lambda_1 \ge n/2$.