S.-T. Yau College Student Mathematics Contests 2016

Probability and Statistics

Individual (5 problems)

Problem 1. A random walker moves on the lattice $\mathbb{Z}^2$ according to the following rule: in the first step it moves to one of its neighbors with probability $1/4$, and then in step $n > 1$ it moves to one of the neighbors that it didn't visit in the step $n-1$ with equal probability. Let T be the time when the random walker steps on a site that it already visited. Please show that the expectation of T is less than 35.

Problem 2. Let X be a $N \times N$ random matrix with i.i.d. random entries, and

$$\mathbb{P}(X_{11} = 1) = \mathbb{P}(X_{11} = -1) = 1/2$$

Define

$$\|X\|_{op} = \sup_{\mathbf{v} \in \mathbb{C}^N : \|\mathbf{v}\|_2 = 1} \|X\mathbf{v}\|_2$$

Please show that for any fixed $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}(\|X\|_{op} \geq N^{1/2+\delta}) = 0$$

Hint: $\|X\|_{op}^2 \leq \mathrm{tr}|X|^2$

Problem 3. Suppose that 2016 balls are put into 2016 boxes with each ball independently being put into box i with probability $\frac{1}{3\times 1008}$ for $i \leq 1008$ and $\frac{2}{3\times 1008}$ for $i > 1008$. Let T be the number of boxes containing exactly 2 balls. Please find the variance of T.

Problem 4. Let $b > a > 0$ be real numbers. Let X be a random variable taking values in $[a, b]$, and let $Y = \frac{1}{X}$. Determine the set of all possible values of $\mathbb{E}(X) \times \mathbb{E}(Y)$.

Problem 5. Let X_1, X_2, $\ldots$ be independent and identically distributed real-valued random variables such that $\mathbb{E}(X_1) = -1$. Let $S_n = X_1 + \cdots + X_n$ for all $n \geq 1$, and let T be the total number of $n \geq 1$ satisfying $S_n \geq 0$. Compute $P(T = \infty)$.