Probability and Statistics

Individual (5 problems)

Problem 1. A random walker moves on the lattice $\mathbb{Z}^2$ according to the following rule: in the first step it moves to one of its neighbors with probability $1/4$, and then in step $n > 1$ it moves to one of the neighbors that it didn’t visit in the step $n − 1$ with equal probability. Let $T$ be the time when the random walker steps on a site that it already visited. Please show that the expectation of $T$ is less than 35.

Problem 2. Let $X$ be a $N \times N$ random matrix with i.i.d. random entries, and

$$P(X_{11} = 1) = P(X_{11} = -1) = 1/2$$

Define

$$\|X\|_{op} = \sup_{v \in \mathbb{C}^N : \|v\|_2 = 1} \|Xv\|_2$$

Please show that for any fixed $\delta > 0$,

$$\lim_{N \to \infty} P(\|X\|_{op} \geq N^{1/2 + \delta}) = 0$$

Hint: $\|X\|^2_{op} \leq \text{tr}|X|^2$

Problem 3. Suppose that 2016 balls are put into 2016 boxes with each ball independently being put into box $i$ with probability $\frac{1}{3 \times 1008}$ for $i \leq 1008$ and $\frac{2}{3 \times 1008}$ for $i > 1008$. Let $T$ be the number of boxes containing exactly 2 balls. Please find the variance of $T$.

Problem 4. Let $b > a > 0$ be real numbers. Let $X$ be a random variable taking values in $[a, b]$, and let $Y = \frac{1}{X}$. Determine the set of all possible values of $E(X) \times E(Y)$.

Problem 5. Let $X_1, X_2, \ldots$ be independent and identically distributed real-valued random variables such that $E(X_1) = -1$. Let $S_n = X_1 + \cdots + X_n$ for all $n \geq 1$, and let $T$ be the total number of $n \geq 1$ satisfying $S_n \geq 0$. Compute $P(T = \infty)$. 