S.-T. Yau College Student Mathematics Contests 2014

Algebra and Number Theory

Individual

This exam of 6 problems is designed to test how much you know rather than how much you don't know. You are not expected to finish all problems but do as much as you can.

Problem 1. Let G be a finite subgroup of $\mathrm{GL}(V)$ where V is an n-dimensional complex vector space.

(a) (5 points) Let

$$H = \{h \in G : hv = \eta(h)v \text{ for some } \eta(h) \in \mathbb{C}^\times \text{ and all } v \in V\}.$$

Prove that H is a normal subgroup of G and that the map $h \mapsto \eta(h)$ is an isomorphism between H and its image in $\mathbb{C}^\times$.

(b) (5 points) Let χ_V be the character function of G acting on V, i.e., $\chi_V(g) = \mathrm{tr}(g)$ with g viewed as an automorphism of V. Prove $|\chi_V(g)| \le n$ for all $g \in G$, and the equality holds if and only if $g \in H$.

(c) (10 points) Let W be an irreducible representation of G. Then W is isomorphic to a direct summand of $V^{\otimes m}$ for some m (as representations of G).

Problem 2. Let $a_1, \dots, a_n$ be nonnegative real numbers.

(a) (6 points) Prove that the $n \times n$ matrix $A = (t^{a_i + a_j})$ is positive semi-definite for every real number $t > 0$. Find the rank of A.

(b) (7 points) Let $B = (c_{ij})_{n\times n}$ be an $n \times n$-matrix with $c_{ij} = \frac{1}{1+a_i+a_j}$. Prove that A is a positive semi-definite matrix.

(c) (7 points) Prove that B is positive definite if and only if a_i are all distinct.

Problem 3. Consider the equations

$$X^2 - 82Y^2 = \pm 2$$

(a) (5 points) Show that if (x, y) is a solution for $X^2 - 82Y^2 = \pm 2$, then $(9x - 82y, x - 9y)$ is a solution for $X^2 - 82Y^2 = \mp 2$.

(b) (7 points) Show that the equations have solutions over $\mathbb{Z}/p^n\mathbb{Z}$ for any n and odd prime p.

(c) (8 points) Show that the equations have no solutions over $\mathbb{Z}$.

Problem 4. Let S and T be nonabelian finite simple groups, and write $G = S \times T$.

(a) (7 points) Show that the total number of normal subgroups of G is four.

(b) (6 points) If S and T are isomorphic, show that G has a maximal proper subgroup not containing either direct factor.

(c) (7 points) If G has a maximal proper subgroup that contains neither of the direct factors of G, show that S and T are isomorphic.

Problem 5. (20 points) Let $\mathbb{F}$ be a finite field and $f_i \in \mathbb{F}[X_1, X_2, ..., X_n]$ be polynomials of degree d_i, where $1 \leq i \leq r$, such that $f_i(0, ..., 0) = 0$ for all i. Show that if

$$n > \sum_{i=1}^{r} d_i,$$

then there exists nonzero solution to the system of equations: $f_i = 0$, for all $1 \leq i \leq r$. (Hint: you may first verify that the number of integral solutions is congruent to the following number modulo p

$$\sum_{X \in \mathbb{F}^n} \prod_{i=1}^{r} \left(1 - f_i(X)^{q-1}\right).$$

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Problem 6.

(a) (5 points) Let A and B be two real $n \times n$ matrices such that $AB = BA$. Show that $\det(A^2 + B^2) \geq 0$.

(b) (15 points) Generalize this to the case of k pairwise commuting matrices.